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INVARIANTS OF TWIST-WISE FLOW EQUIVALENCE

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ABSTRACT. Twist-wise flow equivalence is a natural generalization of flow equivalence that takes account of twisting in the local stable manifold of the orbits of a flow. Here we announce the discovery of two new invariants in this category.

1. FLOW EQUIVALENCE

Square nonnegative integer matrices are used to describe maps on Cantor sets known as subshifts of finite type. Two such matrices are *flow equivalent* if their induced subshifts of finite type give rise to topologically equivalent suspension flows. Here *topologically equivalent* just means there is a homeomorphism, taking orbits to orbits, while preserving the flow direction. A matrix A is *irreducible* if for each (i,j) there is a power n such that the (i,j) entry of A^n is nonzero. In terms of the corresponding subshift and suspension, irreducibility is equivalent to the existence of a dense orbit. Irreducible permutation matrices give rise to flows with a single closed orbit and are thus said to form the *trivial flow equivalence class.* For nontrivial irreducible incidence matrices John Franks has shown that flow equivalence is completely determined by two invariants, the Parry-Sullivan number and Bowen-Franks group. Let A be an $n \times n$ incidence matrix. Then

$$PS(A) = \det(I - A) \quad \text{and} \quad BF(A) = \frac{\mathbb{Z}^n}{(I - A)\mathbb{Z}^n}$$

are the Parry-Sullivan number and the Bowen-Franks group respectively. See [8], [1], and [2] or the recent text [6]. Huang has settled the difficult classification problem arising when the assumption of irreducibility is dropped, [3], [4], [5].

2. TWIST-WISE FLOW EQUIVALENCE

Represent $\mathbb{Z}_2$ by $\{1, t\}$, under multiplication with $t^2 = 1$. Let $A(t)$ be an $n \times n$ matrix with entries of the form $a + bt$, with a and b nonnegative integers. That is A is a matrix over the semigroup ring $\mathbb{Z}^+\mathbb{Z}_2$. Call such a matrix a *twist matrix.* One interpretation of twist matrices is as follows. Suppose the suspension flow for $A(1)$ is realized as a 1-dimensional basic set, $\mathcal{B}$, of saddle type, of a flow on a 3-manifold. For each orbit in $\mathcal{B}$ there is a 2-dimensional local stable manifold, a *ribbon*, if you like. Call the union of such ribbons the *ribbon set*, and denote it by $\mathcal{R}$. Each ribbon

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is either an annulus, a Möbius band, or an infinity long strip. Now, $A(1)$ is the incidence matrix for the first return map ρ on the rectangles of a Markov partition, $\{R_1, \ldots, R_n\}$, of a cross section of a neighborhood of $\mathcal{B}$. Thus, $A_{ij}(1)$ is the number of times R_i passes through R_j. If we orient the rectangles, then we can let a_{ij} be the number of components of $\rho(R_i) \cap R_j$ where orientation is preserved, and b_{ij} be the number of components where orientation is reversed by the action of ρ. Then $A_{ij}(t) = a_{ij} + b_{ij}t$.

It is not necessary that the manifold be 3-dimensional or that there be only one stable eigenvalue. We only need a means of assigning orientations to rectangles of a Markov partition. We note that $A(-1)$ is related to the *structure matrix* of [1]. Two ribbon sets are *topologically equivalent* if there is a homeomorphism between them that preserves the flow direction. This leads us to define two twist matrices to be *twist-wise flow equivalent* if they induce topologically equivalent ribbon sets.

Theorem 2.1. *The numbers $PS(A(\pm 1))$ and the groups $BF(A(\pm 1))$ are invariants of twist-wise flow equivalence.*

It is clear that $PS(A(1))$ and $BF(A(1))$ are invariants in this category. In [9] it is shown that $PS(A(-1))$ is also invariant and it can now be reported that $BF(A(-1))$ is too [10]. We define an additional invariant in §4. However, we still do not possess a complete set of invariants. See §6.

3. Matrix moves

Twist-wise flow equivalence (or *twist equivalence* for short) is generated by three matrix moves [9]. These are called, the *shift* move, the *expansion* move, and the *twist* move, and are denoted by $\overset{s}{\sim}$, $\overset{e}{\sim}$ and $\overset{t}{\sim}$, respectively. The first two generate flow equivalence [8]. We define them below. **Shift:** $A \overset{s}{\sim} B$ if there exist rectangular matrices R and S, over $\mathbb{Z}^+\mathbb{Z}_2$, such that $A = RS$ and $B = SR$. **Expansion:** $A \overset{e}{\sim} B$ if $A = [A_{ij}]$ and

$$B = \begin{bmatrix} 0 & A_{11} & \cdots & A_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & A_{21} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} & \cdots & A_{nn} \end{bmatrix},$$

or vice versa. **Twist:** $A \overset{t}{\sim} B$ if $A = [A_{ij}]$ and

$$B(t) = \begin{bmatrix} A_{11} & tA_{12} & \cdots & tA_{1n} \\ tA_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ tA_{n1} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

The shift move includes relabeling, so the expansion and twist moves can be done on other "locations" in the matrix. See [9] for geometric motivations.

4. The double cover flow

We now consider another means of encoding the twisting of a ribbon set. For a 2-dimensional ribbon set $\mathcal{R}$ place a flow on the boundary with direction parallel to the flow on its core $\mathcal{B}$. Call this the *double cover flow* of $\mathcal{B}$. An incidence matrix

DA can be constructed from a twist matrix $A(t)$ by replacing each entry $a+bt$ with $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. This amounts to using the matrix representation

$$\mathbb{Z}_2 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

As this process is completely formal we need not be restricted to 2-dimensional ribbon sets. It is easy to show that $PS(DA)$ and $BF(DA)$ are invariants of twist equivalence. However, $PS(DA) = PS(A(+1)) \times PS(A(-1))$, and thus contains no new information. But $BF(DA)$ does distinguish twist matrices that the other invariants do not. Yet $PS(A(\pm 1))$, $BF(A(\pm 1))$, and $BF(DA)$ are not complete. See §6.

5. Other Abelian groups

If we replace $\mathbb{Z}_2$ with an arbitrary Abelian group G, the only major change is that the twist move must be replaced by a series of g-moves for each $g \in G$ defined by multiplying the first row by g and the first column by g^{-1}. The Parry-Sullivan invariant becomes an element of the group ring $\mathbb{Z}G$. In [1] the Bowen-Franks group was defined for arbitrary Abelian groups, so there should be no trouble there. If one has a matrix representation of G, then even an analog of the double cover operation should go through. Suppose one has a finite directed graph and an associated subshift of finite type. Let the edges be labeled with elements of G and suppose we seek conjugacies that preserve this additional information. Each closed orbit is paired with the product of its edge labels and this pairing is to be preserved under conjugacies. Then the analogs of the Parry-Sullivan and Bowen-Franks invariants should be useful. We are hopeful that researchers in other areas of dynamics or coding theory will find that they can exploit these ideas.

6. Examples

Consider 2×2 matrices with entries 0, 1, or t, but which are irreducible and nontrivial. For brevity we shall denote $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $abcd$. We will divide these matrices into 6 classes:

- $A = \{1111, 1110, 1101, 1011, 1tt1, 1tt0, 0tt1\}$,
- $B = \{t111, 111t, ttt1, 1ttt\}$,
- $C_1 = \{1t11, 11t1\}$,
- $C_2 = \{tt11, t1t1, 1t1t, 11tt, 1t10, 11t0, 0t11, 01t1, t110, ttt0, 011t, 0ttt\}$,
- $D = \{t11t, tttt, tt10, t1t0, 01tt, 0t1t\}$,
- $E = \{t1tt, tt1t\}$.

Within each of these classes we have shown that the matrices are twist equivalent by constructing the necessary matrix moves. In Table 1 we list the invariants for each class. Classes A and B cannot be distinguished, yet their ribbon sets cannot be homeomorphic since the ribbon set for B contains Möbius bands whereas the closed ribbons for A are all annuli. Hence our invariants are not complete. Of course, one can simply take orientability as an additional invariant. The frustrating point is that the motivation behind all the new invariants was precisely to capture orientation data. The classes C_1 and C_2 also have the same set of invariants. However, we have not been able to tell if they form a single twist class or not.

TABLE 1

Class	PS^+	BF^+	PS^-	BF^-	BF^D
A	-1	0	-1	0	0
B	-1	0	-1	0	0
C_1	-1	0	1	0	0
C_2	-1	0	1	0	0
D	-1	0	3	$\mathbb{Z}_3$	$\mathbb{Z}_3$
E	-1	0	5	$\mathbb{Z}_5$	$\mathbb{Z}_5$

In Table 2 we have listed the invariants for a sampling of 3×3 matrices. Several features stand out. Any finitely generated Abelian group can be realized as a $BF(A(1))$ group [2], and we think this is likely true for $BF(A(-1))$ as well. However, certain groups do not seem to show up as Bowen-Franks groups of double cover flows. For example, we have not found $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ or $\mathbb{Z}_{12}$, though our work here is still very preliminary. In some cases we have $BF^D = BF^+ \oplus BF^-$ (we use a more condensed notation here and in the tables). This never seems to happen if BF^D has infinite order. Does this say anything interesting about the flows?

TABLE 2

Matrix	PS^+	BF^+	PS^-	BF^-	BF^D
$t11111111$	-2	$\mathbb{Z}_2$	-4	$\mathbb{Z}_4$	$\mathbb{Z}_8$
$1t1111111$	-2	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$tt1111111$	-2	$\mathbb{Z}_2$	-2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$t111t111t$	-2	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_3$
$tt11t1111$	-2	$\mathbb{Z}_2$	-4	$\mathbb{Z}_4$	$\mathbb{Z}_8$
$1t1t11111$	-2	$\mathbb{Z}_2$	2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$ttt111111$	-2	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$t11t11111$	-2	$\mathbb{Z}_2$	-2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$tt1tt1111$	-2	$\mathbb{Z}_2$	-2	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$t111t1111$	-2	$\mathbb{Z}_2$	-6	$\mathbb{Z}_6$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
$ttttttttt$	-2	$\mathbb{Z}_2$	4	$\mathbb{Z}_4$	$\mathbb{Z}_8$
$tttttt1tt$	-2	$\mathbb{Z}_2$	6	$\mathbb{Z}_6$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
$tt1ttt1tt$	-2	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_3$
$0t1111111$	-3	$\mathbb{Z}_3$	-1	0	$\mathbb{Z}_3$
$0111t1111$	-3	$\mathbb{Z}_3$	-5	$\mathbb{Z}_5$	$\mathbb{Z}_{15}$
$0t11t1111$	-3	$\mathbb{Z}_3$	-3	$\mathbb{Z}_3$	$\mathbb{Z}_3^2$
$01t1011t1$	-4	$\mathbb{Z}_4$	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$01t10111t$	-4	$\mathbb{Z}_4$	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$011t011t1$	-4	$\mathbb{Z}_4$	-2	$\mathbb{Z}_2$	$\mathbb{Z}_8$
$011t0111t$	-4	$\mathbb{Z}_4$	2	$\mathbb{Z}_2$	$\mathbb{Z}_8$
$01tttt110$	-4	$\mathbb{Z}_4$	6	$\mathbb{Z}_6$	$\mathbb{Z}_{24}$
$011t01110$	-4	$\mathbb{Z}_2^2$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$

The BF groups are presented in Smith Normal Form [7] and many of the calculations were done with the `ismith` command in Maple's `linalg` package. Finally, we present two twist matrices where the Bowen-Franks groups of the double covers is the only distinguishing invariant. Let

$$A(t) = \begin{bmatrix} 3 & 1+t & 2 \\ 1+t & 7 & 1+t \\ 1+t & 1+t & 31 \end{bmatrix} \text{ and } B(t) = \begin{bmatrix} 3 & 1+t & 1+t \\ 1+t & 7 & 1+t \\ 1+t & 1+t & 31 \end{bmatrix}.$$

Then we get

$$PS^+ = -224, \ BF^+ = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{28},$$

$$PS^- = -360, \ BF^- = \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30},$$

for both $A(t)$ and $B(t)$, but

$$BF(DA) = \mathbb{Z}_4 \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{840},$$

while

$$BF(DB) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{840}.$$

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