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THE SCHLÄFLI FORMULA IN EINSTEIN MANIFOLDS WITH BOUNDARY

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ABSTRACT. We give a smooth analogue of the classical Schläfli formula, relating the variation of the volume bounded by a hypersurface moving in a general Einstein manifold and the integral of the variation of the mean curvature. We extend it to variations of the metric in a Riemannian Einstein manifold with boundary, and apply it to Einstein cone-manifolds, to isometric deformations of Euclidean hypersurfaces, and to the rigidity of Ricci-flat manifolds with umbilic boundaries.

RÉSUMÉ. On donne un analogue régulier de la formule classique de Schläfli, reliant la variation du volume borné par une hypersurface se déplaçant dans une variété d'Einstein à l'intégrale de la variation de la courbure moyenne. Puis nous l'étendons aux variations de la métrique à l'intérieur d'une variété d'Einstein riemannienne à bord. On l'applique aux cone-variétés d'Einstein, aux déformations isométriques d'hypersurfaces de E^n, et à la rigidité des variétés Ricci-plates à bord ombilique.

Let M be a Riemannian $(m+1)$-dimensional space-form of constant curvature K, and $(P_t)_{t\in[0,1]}$ a one-parameter family of polyhedra in M bounding compact domains, all having the same combinatorics. Call V_t the volume bounded by P_t, $\theta_{i,t}$ and $W_{i,t}$ the dihedral angle and the $(m-1)$-volume of the codimension 2 face i of P_t. The classical Schläfli formula (see [Mil94] or [Vin93]) is

$$\sum_i W_{i,t}\frac{d\theta_{i,t}}{dt} = mK\frac{dV_t}{dt}. \tag{1}$$

This formula has been extended and used on several occasions recently; see for instance [Hod86], [Bon].

We give a smooth version of this formula, for 1-parameter families of hypersurfaces in (Riemannian of Lorentzian) Einstein manifolds. Then we extend it to variations of an Einstein metric inside a manifold with boundary (a much more general process in dimension above 3). Finally, we give three applications: to the variation of the volume of Einstein cone-manifolds, to isometric deformations of hypersurfaces in the Euclidean space, and to the rigidity of Ricci-flat manifolds with umbilic boundaries. The reader can find the details in [RS98].

Throughout this paper, M is an Einstein manifold of dimension $m+1\geq 3$, and D is its Levi-Civita connection. When dealing with a hypersurface Σ (resp. with

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the boundary ∂M), we call I the *induced metric*, also called the *first fundamental form*, of the corresponding immersion in M. $\overline{D}$ is the Levi-Civita connection of I, B the shape operator, and $I\!I$, $I\!I\!I$ are the second and third fundamental forms of Σ; B and $I\!I$ are defined with respect to an oriented unit normal vector to Σ (resp. to the exterior unit normal to ∂M).

The trace H of B is called the "mean curvature" (some definitions differ by a factor m) and the "higher mean curvatures" $H_k, k \geq 1$, are the higher symmetric functions of the principal curvatures of Σ (resp. ∂M). For instance, $H_2 = (H^2 - \mathrm{tr}(B^2))/2$. dV, dA are the volume elements in M and on Σ (resp. ∂M) respectively.

We denote by δ the divergence acting on symmetric tensors, and by δ^* its formal adjoint. We will often implicitly identify (through the metric) vector fields and 1-forms, as well as quadratic forms and linear morphisms.

1. Deformation of hypersurfaces

Here is an analogue of the Schläfli formula for deformations of (smooth) hypersurfaces in a *fixed* Einstein manifold M, which can be Riemannian or Lorentzian (the other pseudo-Riemannian cases can be treated in the same way).

Theorem 1. *Let Σ be a smooth oriented hypersurface in a (Riemannian) Einstein $(m+1)$-manifold M with scalar curvature S, and v a section of the restriction of TM to Σ. v defines a deformation of Σ in M, which induces variations V', H' and I' of the volume bounded by Σ, mean curvature, and induced metric on Σ. Then:*

$$\frac{S}{m+1}V' = \int_\Sigma \left(H' + \frac{1}{2}\langle I', I\!I\rangle \right) dA. \tag{2}$$

The unit normal used to define H and $I\!I$ should be toward the outside of the volume bounded by Σ. Actually Σ does not need to bound a finite volume domain for this formula to hold. Otherwise, V' is just the derivative of the (signed) volume contained between Σ and Σ_t, for small t. This volume is then oriented by the unit normal to Σ_t used to define H and $I\!I$, which should be toward the exterior.

The proof can be given separately for normal and for tangent deformations of Σ. For tangent deformations, $V' = 0$, and the relation between I' and H' comes from the equation: $\delta I\!I = -dH$, which holds because M is Einstein. For normal deformations, H' is obtained using the trace of $I\!I'$, which is related to the Laplacian of the amplitude of the deformation and to the Ricci curvature of M on the normal to Σ.

The "classical" Schläfli formula (1) for polyhedra in space-forms follows from Theorem 1. Namely, one can apply Theorem 1 to the set of points at distance r on the outside of a convex polyhedron and let $r \to 0$.

In the Lorentzian case, the only difference is that now $g(n,n) = -1$, so the volume variation has a minus sign in the formula. Applying the Lorentzian formula to the set of points at distance ϵ from a polyhedron in S^n_1, one obtains the Schläfli formula for de Sitter polyhedra as in [SP97].

2. Einstein manifolds with boundary

Here $(M, \partial M)$ is a compact manifold with boundary with an Einstein metric g of scalar curvature S. We will prove the same formula as in the previous section, but in a much more general setting: instead of moving a hypersurface in an Einstein manifold, we will be changing the metric (among Einstein metrics of a given scalar curvature) inside this manifold with boundary. Although the two operations are equivalent in dimension at most 3, moving the inside metric is much more general in higher dimensions. On the other hand, our proof only works for Riemannian Einstein manifolds. It is not obvious whether it can be extended to the pseudo-Riemannian setting.

If g is an Einstein metric, we say that a 2-tensor h is an "Einstein variation" of g if the associated variation of the metric induces a variation of the Ricci tensor which is proportional to h, so that $g + \epsilon h$ remains, to the first order, an Einstein manifold with constant scalar curvature.

Theorem 2. *Let h be a smooth Einstein variation of g. Then:*

$$\frac{S}{m+1}V' = \int_{\partial M} H' + \frac{1}{2}\langle h_{|\partial M}, I\!I\rangle dA. \tag{3}$$

As always when studying deformations of Riemannian metrics, the proof needs put some kind of restriction to remove the geometrically trivial deformations, which only correspond to the action of vector fields on the metric. We prevent those deformations in the same way as e.g. in [GL91], [DeT81] or [Biq97], that is, we only consider metric variations h such that $2\delta h + d\,\mathrm{tr}\,h = 0$. The following proposition, which is proved by a fairly simple variational argument, shows that we do not forget any metric variation when doing this.

Proposition 1. *Let h' be a smooth variation of g. Suppose that either $S \leq 0$, or M is strictly convex. There exists another smooth variation h of g such that $2\delta h + d\,\mathrm{tr}(h) = 0$ and $h = h' + \delta^* v_0$, where v_0 is a vector field vanishing on ∂M.*

The variation h of g obtained in Proposition 1 satisfies (because the metric remains Einstein) a simple, elliptic equation. Taking its trace shows that the trace of h also satisfies an elliptic equation which, when integrated over M, expresses V' as some integral over ∂M. A careful examination of this boundary term leads to Theorem 2.

3. Applications

A first application can be found by looking at "singular objects", just as to go from the smooth Theorem 1 to the classical Schläfli formula. There are no polyhedra in general Einstein manifolds, but we can check what happens when we deform Einstein manifolds with cone singularities. It should be pointed out that some of the most interesting modern uses of the classical Schläfli formula concern hyperbolic 3-dimensional cone-manifolds.

Let M be a compact $(m+1)$-manifold, and N a compact codimension 2 submanifold of M. Suppose (g_t) is a 1-parameter family of Einstein metrics with fixed scalar curvature $S \leq 0$ on $M \setminus N$, with a conical singularity on N in the sense that, in normal coordinates around N, g_t has an expansion like

$$g_t = h_t + dr^2 + r^2 d\theta^2 + o(r^2),$$

where h_t is the metric induced on N by g_t, and $\theta \in \mathbf{R}/\alpha_t\mathbf{Z}$ for some $\alpha_t \in \mathbf{R}$. Call V_t the volume of $(M \setminus N, g_t)$, and W_t the volume of (N, h_t). Then

Corollary 1. *V_t varies as: $\frac{S}{m+1}\frac{dV_t}{dt} = W_t \frac{d\alpha_t}{dt}$.*

Note. The same result holds when N has several connected components, each with a different value of α_t. N could also be replaced by a stratified submanifold.

Example. Take $m+1=3$ in the previous example. We find the Schläfli formula for the variation of the volume of a hyperbolic cone-manifold [Hod86].

When the $W_{i,t}$ are constant, the left-hand side of (1) is a polyhedral analogue of the variation of the mean curvature integral of a hypersurface. When $K=0$, the right-hand side is 0. This shows that the "mean curvature" of a 1-parameter family of Euclidean polyhedra with constant induced metric is constant. This was used in [AR97] to prove, using geometric measure theory methods, the following

Theorem 3. *In $\mathbf{R}^{m+1}$, the integral of the mean curvature remains constant in an isometric deformation of a hypersurface.*

This theorem follows immediately from our Theorem 1.

On the other hand, the integral mean curvature is **not** determined by the metric on ∂M: this is already visible in $\mathbf{R}^3$. Namely, some metrics on S^2 admit two isometric embeddings in $\mathbf{R}^3$: the classical example is that a (topological) sphere in $\mathbf{R}^3$ which is tangent to a plane along a circle can be "flipped" so as to obtain another embedding with the same induced metric [Spi75]. Those two embeddings do not in general have the same integral mean curvature—and thus we have a complicated way of seeing that the two flipped surfaces cannot be bent one into the other.

Formula (3) is even simpler for variations which vanish on ∂M:

Theorem 4. *If h is a smooth Einstein variation of g which does not change the induced metric on ∂M, then*

$$\int_{\partial M} H' dV = \frac{S}{m+1} V'.$$

In particular, for $S=0$, this implies that the integral of the mean curvature of the boundary is constant under an Einstein variation which does not change the induced metric on ∂M; this is a direct generalization of Theorem 3.

The analogue of Theorem 3 is also true, but in a pointwise sense, for the higher mean curvatures:

Theorem 5. *In $\mathbf{R}^{m+1}$, the integral of H_k $(k \geq 2)$ remains constant in an isometric deformation of a hypersurface.*

This comes from the following (probably classical) description of the possible isometric deformations of a hypersurface for $m+1 \geq 4$:

Remark 1. Let $(\Sigma_t)_{t\in[0,1]}$ be a 1-parameter family of hypersurfaces in a space-form, such that the induced metric I_t is constant to the first order at $t=0$. Then, at each point, one of the following is true:

- $I\!I_0 = 0$;
- $\mathrm{rk}(I\!I_0) \leq 2$, and $I\!I'_0$ vanishes on the kernel of $I\!I_0$;
- $I\!I'_0 = 0$;

where $I\!I_t$ is the second fundamental form of Σ_t, and $I\!I'_t$ its variation.

Theorem 5 clearly follows, because H'_k is zero for $k \geq 3$ in each case, and the Gauss formula gives the proof for $k=2$.

Denote by Σ^ϵ_t the parallel surface at distance ϵ from Σ_t. It is well known (see, e.g., Santalo's book [San76]) that the area of Σ^ϵ is a polynomial in ϵ where the coefficent of ϵ^k is (essentially) the k-th mean curvature of Σ. The two Theorems 3 and 5 can then be combined as stating that:

Theorem 6. *The area of Σ_t^ϵ stays constant when Σ_t is a bending of Σ_0.*

Finally, we use the Schläfli formula above to prove a rigidity result for Ricci-flat manifolds with umbilic boundary; it is a generalization of the classical result (see [Spi75]) that the round sphere is rigid in $\mathbf{R}^3$, that is, it cannot be deformed smoothly without changing its induced metric.

Lemma 1. *Suppose $(M, \partial M)$ is a compact $(m+1)$-manifold with boundary, and $(h_t)_{t\in[0,1]}$ is a nontrivial 1-parameter family of Ricci-flat metrics on M inducing the same metric on ∂M, and such that ∂M is umbilic and convex (or concave) for h_0. Then ∂M has at least two connected components, and (h_t) corresponds to the displacement of some connected component(s) of ∂M under the flow of some Killing field(s) of M.*

This kind of rigidity result could be used in the future to prove that, given a Ricci-flat manifold M with umbilic boundary and induced metric g_0 on the boundary, any metric close to g_0 on ∂M can be realized as induced on ∂M by some Ricci-flat metric on M. In this setting, rigidity corresponds to the local injectivity of an operator sending the metrics on M to the metrics on ∂M. In dimension 3, this would be a part of the classical result (see [Nir53]) that metrics with curvature $K > 0$ on S^2 can be realized as induced by immersions into $\mathbf{R}^3$. This circle of ideas is illustrated in [Sch98].

The first point is to understand what an umbilic hypersurface in an Einstein manifold is. By definition, if N is a Riemannian manifold and S is a hypersurface, then S is umbilic if, at each point $s \in S$, $I\!I$ is proportional to I, with a proportionality constant $\lambda(s)$ depending on s. Now

Remark 2. If N is Einstein, then λ is constant on each connected component of S.

The next step is an inequality concerning the integral of the mean curvature squared.

Proposition 2. *Let (M, g) be an Einstein manifold with boundary, with scalar curvature S. Call $\overline{S}$ the scalar curvature of $(\partial M, g_{|\partial M})$. Then the mean curvature $H = \mathrm{tr}(I\!I)$ of ∂M satisfies*

$$\frac{\overline{S}}{m-1} - \frac{S}{m+1} \leq \frac{H^2}{m}$$

with equality if and only if ∂M is umbilic.

Proposition 2 and Theorem 4 lead to the proof of Lemma 1. The key point is that by Proposition 2, H^2 is pointwise minimal when the boundary is umbilic, while Theorem 4 shows that the integral of H is constant. The boundary therefore has to remain umbilic in an Einstein variation which vanishes on the boundary.

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