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CELLULAR ALGEBRAS AND QUASI-HEREDITARY ALGEBRAS: A COMPARISON

STEFFEN KÖNIG AND CHANGCHANG XI

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Abstract. Cellular algebras have been defined in a computational way by the existence of a special kind of basis. We compare them with quasi-hereditary algebras, which are known to carry much homological and categorical structure. Among the properties to be discussed here are characterizations of quasi-hereditary algebras inside the class of cellular algebras in terms of vanishing of cohomology and in terms of positivity of the Cartan determinant.

1. Introduction

To a large extent, algebraic representation theory of Lie algebras, algebraic groups and related finite groups deals with finite-dimensional algebras which are cellular [6] or quasi-hereditary [3]. Group algebras of symmetric groups and their Hecke algebras are known to be cellular as well as various generalizations (e.g. Brauer algebras, cyclotomic Hecke algebras, Temperley–Lieb algebras, partition algebras). Several of these algebras also have been used in other contexts like topology (invariants of knots or manifolds) or statistical mechanics. Schur algebras associated with semisimple algebraic groups in any characteristic and blocks of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated with semisimple complex Lie algebras are cellular as well, but they also satisfy the stronger condition to be quasi-hereditary. A quasi-hereditary structure comes both with desirable numerical properties (decomposition matrices are square matrices, the number of simple modules can be read off from a defining chain of ideals) and with homological structure (finite global dimension, vanishing results on certain cohomology groups, stratification of derived module categories, existence of 'tilting modules' and derived equivalences, possibility to define 'Kazhdan–Lusztig' theory), and also there is a categorical definition (which cannot exist for cellular algebras; see below). Many cellular algebras, in particular Brauer algebras and partition algebras, are known to be quasi-hereditary for some choice of parameters and not quasi-hereditary for some other choice (typically 'at zero').

In contrast to quasi-hereditary algebras, whose definition already comes with a lot of structure, cellular algebras have been defined first in a purely computational

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way, by requiring the existence of a basis with nice multiplicative properties. However, recently a theory has emerged which discusses homological and categorical structures in this class of algebras. In particular, this theory clarifies the relation with quasi-hereditary algebras both from the abstract point of view and from that of checking examples in practice. The aim of this note is to survey this development.

2. Definitions

From now on, by an algebra we always mean an associative algebra, which is finite dimensional over a field k.

The original definition of cellular algebras is as follows.

Definition 2.1 (Graham and Lehrer, [6]). An associative k-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $C^{\lambda}_{S,T}$, where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k-linear anti-automorphism of A with $i^2 = \mathrm{id}$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T}) + r'$, where r' is a linear combination of basis elements with upper index μ strictly less than λ, and where the coefficients $r_a(U,S) \in k$ do not depend on T.

In the following we shall call a k-linear anti-automorphism i of A with $i^2 = \mathrm{id}$ an involution of A. In [7] it has been shown that this definition is equivalent to the following one.

Definition 2.2 ([7]). Let A be a k-algebra. Assume there is an anti-automorphism i on A with $i^2 = \mathrm{id}$. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ has finite k-dimension and that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_k i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_k i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_k i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$), the quotient $J'_j = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The first definition can be used to check concrete examples. The second definition, however, is often more handy for theoretical and structural purposes.

Typical examples of cellular algebras are the following: Group algebras of symmetric groups, or more general Hecke algebras of type A or even of Ariki–Koike type (i.e. cyclotomic Hecke algebras) [6], Brauer algebras of types B and C [6] (see also

[10] for another proof), partition algebras [16] and various kinds of Temperley–Lieb algebras [6].

Let us also recall the definition of quasi-hereditary algebras introduced in [3].

Definition 2.3 (Cline, Parshall, and Scott [3]). Let A be a k-algebra. An ideal J in A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}(A))J = 0$ and J is a projective left (or right) A-module. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.

Examples of quasi-hereditary algebras are blocks of category $\mathcal{O}$ [1] and Schur algebras [14, 5]. The precise relation to highest weight categories is described in [3].

3. Results

Being interested in structural results, the first question one has to ask is that of Morita invariance. In the case of quasi-hereditary algebras, Cline, Parshall, and Scott [3] proved the equivalence of the definition of quasi-hereditary algebras given above with another one, which is in terms of 'highest weight categories' and hence categorical. Thus Morita invariance follows immediately. For cellular algebras the situation is more delicate.

Theorem 3.1 ([8]). *Let k be a field of characteristic different from two. Then the notion of 'cellular algebra' over k is Morita invariant. This is not true over fields of characteristic two.*

Often, cellular structures can be defined in a characteristic free way, e.g. for integral group rings of symmetric groups. The second part of the theorem says that, in general, it is impossible to transfer a cellular structure to a Morita equivalent algebra (unless two is invertible in the ground ring). In particular, there cannot exist a purely categorical definition of cellular algebras.

For a quasi-hereditary algebra A, the length of a longest heredity chain equals the number of isomorphism classes of simple A-modules. Conversely, a cellular algebra with a cell chain of this length must be quasi-hereditary. However, if the algebra is cellular, but not quasi-hereditary, then the length of a cell chain is not an invariant of the algebra any more. An example (of a local algebra of dimension fourteen) is given in [9].

In the representation theory of finite groups or related topics one often uses the following hierarchy of finite-dimensional algebras:

$$\begin{array}{c} \{semisimple\} \\ \cap \\ \{symmetric\} \\ \cap \\ \{weakly\ \ symmetric\} \\ \cap \\ \{quasi\text{-}Frobenius = self\text{-}injective\} \end{array}$$

Here self-injective means that each projective module is injective as well, whereas weakly symmetric says that the projective cover of any given simple module is the injective envelope of the same simple module. That is, the permutation $\mathrm{top}(P) \mapsto$

soc(P) (for P indecomposable projective–injective) is the identity. It is well known that in general all these inclusions are proper. Within the class of cellular algebras, however, one inclusion is the identity.

Theorem 3.2 ([12]). *Let A be a cellular algebra. If A is self-injective, then it is weakly symmetric.*

The main results we have obtained deal with the question when a cellular algebra is quasi-hereditary. This question splits into the following problems.

Problem 1. How to characterize the quasi-hereditary algebras among the cellular ones by a structural property?

Problem 2. How to characterize the quasi-hereditary algebras among the cellular ones by a numerical property?

Problem 3. Given a cellular algebra with a cell chain of ideals, how to decide whether it is quasi-hereditary?

The equivalence of (a) and (c) in the following theorem answers the first problem. Problem 2 is solved by the equivalence between (a) and (d). And the equivalence between non-(a) and non-(b) tells us how to prove that a cellular algebra is not quasi-hereditary, i.e. how to solve Problem 3.

Theorem 3.3 ([11]). *Let k be a field and A a cellular k-algebra (with respect to an involution i). Then the following are equivalent:*

(a) Some cell chain of A (with respect to some involution, possibly different from i) is a heredity chain as well, i.e. it makes A into a quasi-hereditary algebra.

(a′) There is a cell chain of A (with respect to some involution, possibly different from i) whose length equals the number of isomorphism classes of simple A-modules.

(b) Any cell chain of A (with respect to any involution) is a heredity chain.

(c) The algebra A has finite global dimension; i.e., there exists an $N \in \mathbb{N}$ such that $Ext^i_A(X,Y) = 0$ for all $i \geq N$ and for all A-modules X and Y.

(d) The Cartan matrix (recording the composition multiplicities of simple modules in indecomposable projective modules) of A has determinant one.

The determinant of the Cartan matrix of a cellular algebra always is a positive integer [11].

The main open cases of cellular algebras to be checked for quasi-heredity were Brauer algebras [15, 6] and partition algebras [13].

Problem 4. Determine precisely for which choice of parameters a Brauer algebra or a partition algebra is quasi-hereditary.

Problem 4 is answered by the next two results which apply the equivalence of (a) and (b) in Theorem 3.3.

Theorem 3.4 ([11]). *Let k be any field, fix $\delta \in k$, and denote by $B(r,\delta)$ the Brauer algebra on $2r$ vertices and with parameter δ.*

Then $B(r,\delta)$ is quasi-hereditary if and only if

(1) δ is not zero or r is odd; and

(2) the characteristic of k is either zero or strictly greater than r.

This extends previous results by Graham and Lehrer [6]; they proved the 'if'-part in [6], 4.16. and 4.17.

For partition algebras we have

Theorem 3.5 ([11]). *Let k be any field, fix $\delta \in k$, and denote by $P(r, \delta)$ the partition algebra on $2r$ vertices and with parameter δ.*

Then $P(r, \delta)$ is quasi-hereditary if and only if δ is not zero and the characteristic of k is either zero or strictly greater than r.

Martin [13] had shown this in case of characteristic zero and $\delta \neq 0$. In [16], the 'if'-part of Theorem 3.5 is proved.

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: `koenig@mathematik.uni-bielefeld.de`

Department of Mathematics, Beijing Normal University, 100875 Beijing, P. R. China

E-mail address: `xicc@bnu.edu.cn`