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THE FIRST EIGENVALUE OF A RIEMANN SURFACE

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ABSTRACT. We present a collection of results whose central theme is that the phenomenon of the first eigenvalue of the Laplacian being large is typical for Riemann surfaces. Our main analytic tool is a method for studying how the hyperbolic metric on a Riemann surface behaves under compactification of the surface. We make the notion of picking a Riemann surface at random by modeling this process on the process of picking a random 3-regular graph. With this model, we show that there are positive constants C_1 and C_2 independent of the genus, such that with probability at least C_1, a randomly picked surface has first eigenvalue at least C_2.

In this note, we announce a collection of results ([10, 11, 12]) connected to the behavior of the first eigenvalue $\lambda_1(S)$ of a compact Riemann surface of large genus, endowed with a metric of constant curvature -1. These results have as their common theme that the phenomenon of λ_1 large is in some sense typical. To make the notion of "typical" precise, we model the process of picking a Riemann surface at random on the process of picking a 3-regular graph at random.

The idea of studying the first eigenvalue of a Riemann surface via the study of eigenvalues of 3-regular graphs comes from the work of Buser [13, 14]. In effect, our approach here is a variation on his idea, where we first study the behavior of λ_1 on finite-area Riemann surfaces connected to 3-regular graphs, and then see how λ_1 changes when we compactify the surface.

Our main analytic tool is a method for studying how the hyperbolic metric of a finite-area Riemann surface behaves under such a compactification. This method was introduced in [6], and is based on the Ahlfors-Schwarz Lemma ([1]; see also [5]).

We then have:

Theorem 1 ([10]). *For all ϵ, there exists N such that, for $g \geq N$, there is a compact Riemann surface S_g of genus g satisfying*

$$\lambda_1(S_g) \geq \frac{171}{784} - \epsilon.$$

The number 171/784 comes from the improvement by Luo, Rudnick, and Sarnak [17] of the Selberg 3/16 Theorem [18]. If Selberg's conjecture were true, we would be able to replace 3/16 by 1/4, the best possible value. More generally, Theorem 1

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contains a method to build from a collection of Riemann surfaces $\{S_i\}$ of large first eigenvalue a larger collection of surfaces whose genera may include all but finitely many genera, whose first eigenvalues satisfy a similar bound.

To state our next results, let $\mathcal{F}_{n,k}$ denote the set of k-regular graphs on n vertices, and $\mathcal{F}^*_{n,k}$ the set of pairs $(\Gamma, \mathcal{O})$, where $\Gamma \in \mathcal{F}_{n,k}$ and $\mathcal{O}$ is an orientation on Γ —that is, for each vertex v of Γ, $\mathcal{O}$ prescribes a cyclic ordering of the edges emanating from $\mathcal{O}$. Let $\mathcal{F}^!_{n,k}$ denote the subset of $\mathcal{F}_{n,k}$ consisting of graphs without loops, double edges, or 3-cycles. As a probability space, $\mathcal{F}^!_{n,k}$ has positive measure in $\mathcal{F}_{n,k}$ bounded away from 0 as $n \to \infty$.

We describe a way of associating to the pair $(\Gamma, \mathcal{O}) \in \mathcal{F}_{n,3}$ a compact Riemann surface $S^C(\Gamma, \mathcal{O})$. By a theorem of Belyi [2], the surfaces that arise in this way are dense in the space of all compact Riemann surfaces.

We then have:

Theorem 2 ([11]). *There exists a constant C_1 with the following property:*

(a) *If Γ is picked randomly from $\mathcal{F}^!_{n,4}$ for n odd, then with probability $\to 1$ as $n \to \infty$, there is an orientation $\mathcal{O}$ on Γ so that for any splitting of $(\Gamma, \mathcal{O})$ to a 3-regular graph $(\Gamma', \mathcal{O}')$, the surface $S^C(\Gamma', \mathcal{O}')$ satisfies:*
   (i) *$S^C(\Gamma', \mathcal{O}')$ has genus $\frac{n+1}{2}$;*
   (ii) *$\lambda_1(S^C(\Gamma', \mathcal{O}')) \geq C_1$.*

(b) *If Γ is picked randomly from $\mathcal{F}^!_{n,3}$ with $n \equiv 2 \,(\mathrm{mod}\, 4)$, then with probability $\to$ 1 as $n \to \infty$, there is an orientation $\mathcal{O}$ on Γ such that the surface $S^C(\Gamma', \mathcal{O}')$ satisfies:*
   (i) *$S^C(\Gamma', \mathcal{O}')$ has genus $\frac{n+2}{4}$;*
   (ii) *$\lambda_1(S^C(\Gamma', \mathcal{O}')) \geq C_1$.*

Theorem 3 ([12]). *There exist constants C_2, C_3, C_4, and C_5 with the following property: if $(\Gamma, \mathcal{O})$ is picked randomly from $\mathcal{F}^*_{n,3}$, then, as $n \to \infty$, the surface $S^C(\Gamma, \mathcal{O})$ will have the following properties, with probability at least C_2:*

(a) *$\lambda_1(S^C(\Gamma, \mathcal{O})) \geq C_3$.*

(b) *The length of the shortest geodesic* $\mathrm{syst}(S^C(\Gamma, \mathcal{O}))$ *of $S^C(\Gamma, \mathcal{O})$ satisfies*

$$\mathrm{syst}(S^C(\Gamma, \mathcal{O})) \geq C_4.$$

(c) *The diameter* $\mathrm{diam}(S^C(\Gamma, \mathcal{O}))$ *satisfies*

$$\mathrm{diam}(S^C(\Gamma, \mathcal{O})) \leq C_5 \log(\mathrm{genus}(S^C(\Gamma, \mathcal{O}))).$$

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1. Open and closed Riemann surfaces

Let S^O be an open Riemann surface, carrying a complete metric of constant curvature -1 and finite area. Then there is a unique compactification S^C of S^O whose conformal structure is uniquely determined from S^O. In general, S^C need not carry a hyperbolic metric, but under favorable circumstances it will, and indeed the hyperbolic metrics on S^O and S^C will be closely related.

Theorem 4 ([6])**.** *For all ϵ, there exists L such that, if all the cusps of S^O have length $\geq L$, then there are canonically defined cusp neighborhoods $\{\mathcal{U}_i\}$ on S^O and S^C such that the hyperbolic metrics ds_O^2 and ds_C^2 satisfy*

$$\frac{1}{(1+\epsilon)} ds_O^2 \leq ds_C^2 \leq (1+\epsilon) ds_O^2$$

outside the sets $\{\mathcal{U}_i\}$.

See [6] for a precise statement. The length of a cusp is the length of the longest closed horocycle about the cusp.

The inverse process was described in [10]:

Theorem 5 ([10])**.** *Given L, there exisis a number R with the following property: If S is a compact Riemann surface, and $\{p_1, \ldots, p_k\}$ points on S such that*

(a) *The injectivity radius about each point is at least R, and*
(b) *The balls $B(p_i, R)$ of radius R about the points p_i are pairwise disjoint,*

then $S - \{p_1, \ldots, p_k\}$ has cusps of length $\geq L$.

Using this, one shows:

Theorem 6 ([6, 10])**.** *For L sufficiently large, there is a constant $C(L)$ such that:*

(i) *The Cheeger constants $h(S^O)$ and $h(S^C)$ satisfy:*

$$\frac{1}{C(L)} h(S^O) \leq h(S^C) \leq C(L) h(S^O).$$

(ii) *The first eigenvalues $\lambda_1(S^O)$ and $\lambda_1(S^C)$ satisfy*

$$\frac{1}{C(L)} \lambda_1(S^O) \leq \lambda_1(S^C) \leq C(L) \lambda_1(S^O).$$

(iii) *The shortest closed geodesics satisfy*

$$\frac{1}{C(L)} \operatorname{syst}(S^O) \leq \operatorname{syst}(S^C) \leq C(L) \operatorname{syst}(S^O).$$

Furthermore, $C(L) \to 1$ as $L \to \infty$.

By combining this result with the technique of [15] of closing off cusps by forming handles, we prove

Theorem 7 ([10])**.** *Let $\{S_i\}$ be a collection of compact Riemann surfaces with the following properties:*

(a) *There exists $\lambda > 0$ such that $\lambda_1(S_i) > \lambda$ for all i.*
(b) $\operatorname{syst}(S_i) \to \infty$ *as $i \to \infty$.*
(c) *For every $C > 1$ there exists an N such that*

$$\{x \in \mathbb{R} : x > N\} \subset \bigcup_i [1, C](\text{genus}(S_i)).$$

Then, for all ϵ, there exists N such that for $g \geq N$ there exists a surface S_g of genus g with

$$\lambda_1(S_g) \geq \lambda - \epsilon.$$

Applying this theorem to the compactifications of the modular surfaces then gives Theorem 1.

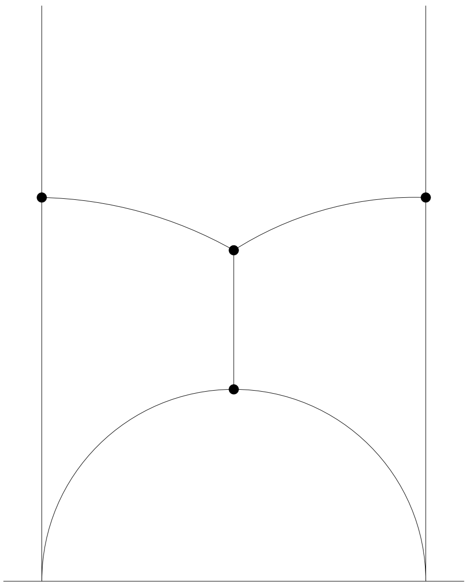

FIGURE 1. The marked ideal triangle T.

2. Riemann surfaces and 3-regular graphs

Let Γ be a 3-regular graph. An *orientation* $\mathcal{O}$ of Γ is an assignment to each vertex v of Γ of a cyclic ordering of the edges emanating from v. It is clear that a 3-regular graph on n vertices possesses 2^n orientations.

To the pair $(\Gamma, \mathcal{O})$ we will assign two surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$. The surface $S^O(\Gamma, \mathcal{O})$ is obtained by gluing one copy of the ideal hyperbolic triangle T shown in Figure 1 for each vertex of Γ, such that the natural orientation of the geodesic segments on T matches up with the orientation about the vertex. Whenever two vertices are joined by an edge, we glue the corresponding triangles together, subject to the following conditions:

(i) The geodesic segments on the copies of T are glued together.
(ii) The orientations on the copies of T (as complex manifolds with boundary) are preserved.

The surfaces $S^C(\Gamma, \mathcal{O})$ are the compactifications of the surfaces $S^O(\Gamma, \mathcal{O})$.

As discussed in [8], the geometry and even the topology of the surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$ depend very strongly on the orientation $\mathcal{O}$. We will say that a path γ on $(\Gamma, \mathcal{O})$ is a left-hand-turn path if, whenever it arrives at a vertex, it turns left according to the orientation $\mathcal{O}$. Then each cusp of $S^O(\Gamma, \mathcal{O})$ is associated to a unique left-hand-turn path. If we denote by $\#(LHT)$ the number of these paths, then the genus of $S^C(\Gamma, \mathcal{O})$ is clearly

$$\text{genus}(S^C(\Gamma, \mathcal{O})) = 1 + \frac{n - 2\#(LHT)}{4}.$$

The length of the cusp corresponding to a given left-hand-turn path γ is precisely the number of edges in γ.

Many geometric properties of the surface $S^O(\Gamma, \mathcal{O})$ are reflected in the pair $(\Gamma, \mathcal{O})$. Some of these properties follow from general properties of covering manifolds ([7] and [9]). Other properties depend more delicately on this particular construction. When the graph has no short left-hand-turn paths, then these properties descend to properties on $S^C(\Gamma, \mathcal{O})$ via Theorem 6.

Theorem 8. *For some L sufficiently large, there are constants $C_1, \ldots, C_6$ with the following property: Suppose that $(\Gamma, \mathcal{O})$ has no left-hand-turn paths of length $\leq L$. Then*

(i) $C_1\lambda_1(\Gamma) \leq \lambda_1(S^C(\Gamma, \mathcal{O})) \leq C_2\lambda_1(\Gamma)$.
(ii) $C_3 h(\Gamma) \leq S^C(\Gamma, \mathcal{O}) \leq C_4 h(\Gamma)$.
(iii) $\log(\operatorname{syst}(\Gamma)) \leq \operatorname{syst}(S^C(\Gamma, \mathcal{O})) \leq C_5 \operatorname{syst}(\Gamma)$.
(iv) ([11]) $\operatorname{diam}(S^C(\Gamma, \mathcal{O})) \leq C_6 \operatorname{diam}(\Gamma)$.

With such strong control over the surface $S^C(\Gamma, \mathcal{O})$, one might be led to expect that the surfaces $S^C(\Gamma, \mathcal{O})$ are rather rare. It is therefore rather surprising that they in fact are quite common.

Theorem 9 ([2, 12])**.** *Given any compact Riemann surface S, there are arbitrarily small deformations S_ϵ of S such that $S_\epsilon = S^C(\Gamma, \mathcal{O})$ for some pair $(\Gamma, \mathcal{O})$.*

3. Models of random graphs

Theorems 2 and 3 are now obtained by an analysis of the process of picking a random graph. To carry out this analysis, we make use of the model of random graphs considered by Bollobás [3, 4]. In this model, a k-regular graph on n vertices is constructed at random by putting nk balls into a hat, k balls for each vertex. The balls are drawn out of the hat in pairs, and an edge drawn between v_1 and v_2 each time a pair of balls corresponding to v_1 and v_2 is drawn. An orientation on the graph may be determined by the order in which the corresponding pairs are drawn.

We will need the following results of [3] and [4]:

Theorem 10. (i) ([4]) *There is a constant C_1 such that, as $n \to \infty$, the probability that $H(\Gamma) \geq C_1$ tends to 1.*
(ii) ([3]) *Let $X_1, \ldots, X_L$ denote the random variable*

$$X_j = \text{ the number of closed paths of length } j \text{ in } \Gamma.$$

Then, for L fixed and $n \to \infty$, the variables $X_1, \ldots, X_L$ tend to independent Poisson distributions.

To establish Theorem 2 (a), we seek the probability that a randomly chosen 4-regular graph will have an orientation with precisely one left-hand-turn path. Using ideas of [20, 21], it is shown in [11] that this will happen with probability $\to 1$ as long as Γ has no closed loops of length 1. The proof of Theorem 2 (b) is similar, using [19] in place of [20, 21].

To establish Theorem 3, we estimate the probability that the pair $(\Gamma, \mathcal{O})$ has no left-hand-turn paths of length $\leq L$. This will certainly be the case if it has no closed paths of length $\leq L$ whatsoever, from which Theorem 3 follows. By refining this argument, we may get substantially better estimates for the constant C_2.

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