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FAMILY ALGEBRAS

A. A. KIRILLOV

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Abstract. A new class of associative algebras is introduced and studied. These algebras are related to simple complex Lie algebras (or root systems). Roughly speaking, they are finite dimensional approximations to the enveloping algebra $U(\mathfrak{g})$ viewed as a module over its center.

It seems that several important questions on semisimple algebras and their representations can be formulated, studied and sometimes solved in terms of our algebras.

Here we only start this program and hope that it will be continued and developed.

0. Introduction

The aim of this paper is to introduce and study a new class of associative algebras: the so-called family algebras. We assume that the reader is acquainted with the general background of the theory of semisimple Lie algebras (see e.g. [H]).

Let $\mathfrak{g}$ be a simple complex Lie algebra with the canonical decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

We denote by P (resp. Q) the weight (resp. root) lattice in $\mathfrak{h}^*$ and by P_+ (resp. Q_+) the semigroup generated by fundamental weights $\omega_1, \omega_2, \ldots, \omega_l$ (resp. by simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$).

For every $\lambda \in P_+$ let (π_λ, V_λ) be an irreducible representation of $\mathfrak{g}$ with highest weight λ. We denote by $d(\lambda)$ the dimension of V_λ.

Let λ^* denote the highest weight of the dual (or contragredient) representation which acts in V_λ^* by $\pi_{\lambda^*}(X) = -(\pi_\lambda(X))^*$. It is clear that $d(\lambda) = d(\lambda^*)$.

The space $\operatorname{End} V_\lambda$ is isomorphic to the matrix space $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})$ and has a $\mathfrak{g}$-module structure defined by

$$X \cdot A = [\pi_\lambda(X), A].$$

Recall that the symmetric algebra $S(\mathfrak{g})$ and the enveloping algebra $U(\mathfrak{g})$ also have (isomorphic) $\mathfrak{g}$-module structures.

Let G be a connected and simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. The action of $\mathfrak{g}$ on $\operatorname{End} V_\lambda$, $S(\mathfrak{g})$ and $U(\mathfrak{g})$ gives rise to the corresponding action of

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G. We define two kinds of **family algebras**: the **classical** algebra $\mathcal{C}_\lambda(\mathfrak{g})$ and the **quantum** algebra $\mathcal{Q}_\lambda(\mathfrak{g})$ by

$$\mathcal{C}_\lambda(\mathfrak{g}) := (\operatorname{End} V_\lambda \otimes S(\mathfrak{g}))^G, \qquad \mathcal{Q}_\lambda(\mathfrak{g}) := (\operatorname{End} V_\lambda \otimes U(\mathfrak{g}))^G. \tag{1}$$

We hope to apply the theory of family algebras to several important questions on semisimple algebras and their representations.

Here we only start this program and formulate some preliminary results.

1. Generalities about family algebras

1.1. Main definitions. First of all, we make the basic definition (1) more visual and practical. For a given pair $(\mathfrak{g}, \lambda)$ let us consider the set of all matrices A of order $d(\lambda)$ with elements from $S(\mathfrak{g})$ or from $U(\mathfrak{g})$.

We can define two different actions of an element $g \in G$ on A:

— the right action via conjugation by the matrix $\pi(g)$:

$$A \mapsto A \cdot g := \pi(g)^{-1} A \pi(g);$$

— the left action by application of $\operatorname{Ad}(g)$ to all matrix elements of A:

$$A \mapsto g \cdot A, \qquad \text{where} \qquad (g \cdot A)_{ij} = \operatorname{Ad}(g) A_{ij}.$$

The statement that for any $g \in G$ the results of these two actions on A coincide (i.e. $A \cdot g = g \cdot A$) is equivalent to the claim that A belongs to the family algebra.

The infinitesimal version of this condition is

$$[A,\, \pi(X)]_{ij} = \operatorname{ad} X(A_{ij}). \tag{2}$$

For the classical family algebras the condition (2) can be rewritten in terms of the canonical Poisson structure on $\mathfrak{g}^*$. For this end we identify $S(\mathfrak{g})$ with $\operatorname{Pol}(\mathfrak{g}^*)$ and consider the basic elements $X_1, \dots, X_n$ of $\mathfrak{g}$ as coordinates on $\mathfrak{g}^*$. By ∂^i we denote the partial derivative with respect to X_i. Then the Poisson bracket has the form

$$\{f_1,\, f_2\} = c_{ij}^k\, X_k\, \partial^i f_1\, \partial^j f_2.$$

The condition (2) is equivalent to

$$[A,\, \pi(X)]_{ij} = \{X,\, A_{ij}\} \qquad \text{or simply} \qquad [A,\, \pi(X)] = \{X,\, A\}. \tag{2'}$$

We shall use yet another interpretation of the definition of classical family algebras. Consider elements of $\mathcal{C}_\lambda(\mathfrak{g})$ as polynomial matrix-valued functions on $\mathfrak{g}^*$. Then the condition that a function A corresponds to an element of the family algebra means

$$A(K(g)F) = \pi_\lambda(g) A(F) \pi_\lambda(g)^{-1}. \tag{2''}$$

The useful corollary of this definition is

Theorem S. *Assume that π_λ has a simple spectrum (i.e. all weights have multiplicity 1). Then $\mathcal{C}_\lambda(\mathfrak{g})$ is commutative.*

Proof. It is enough to check the commutativity of $A(F)$ and $B(F)$ for generic F. Since a generic element of $\mathfrak{g}^* \cong \mathfrak{g}$ is conjugate to an element of the Cartan subalgebra, we can assume that $F \in \mathfrak{h}$. But then (2″) implies that the values of $A(F)$ and $B(F)$ are diagonal matrices, hence they commute. □

Now we recall some properties of Lie algebras $\mathfrak{g}$ which possess an $\mathrm{Ad}(G)$-invariant symmetric bilinear form $(\cdot\,,\,\cdot)$. For a simple Lie algebra $\mathfrak{g}$ such a form always exists and is unique up to a scalar factor. We will usually use the form

$$(X,\,Y) = \mathrm{tr}\,(\pi(X)\pi(Y)), \tag{3}$$

where π is a non-trivial irreducible representation of $\mathfrak{g}$ of minimal dimension.

The $\mathrm{Ad}(G)$-invariant form allows one to identify adjoint and coadjoint modules and thus identify $S(\mathfrak{g})$ not only with the polynomial algebra $P(\mathfrak{g}^*)$ but also with the algebra $S(\mathfrak{g}^*)$, which we regard as the algebra of differential operators on $\mathfrak{g}^*$ with constant coefficients. We denote by $\Delta(P)$ the differential operator on $\mathfrak{g}^*$ associated with $P \in S(g)$.

In particular, if $\{X_i\}$ and $\{X^i\}$ are any dual bases in $\mathfrak{g}$ with respect to the chosen Ad-invariant bilinear form on $\mathfrak{g}$, then we have

$$\Delta(X^i) = \partial^i \qquad \text{and} \qquad X_i\Delta(X^i) = E \text{ (the Euler operator)}. \tag{4}$$

The form $(\cdot\,,\,\cdot)$ extends to a non-degenerate $\mathrm{Ad}(G)$-invariant form on $S(\mathfrak{g}) \cong P(\mathfrak{g}^*)$ (also denoted by parentheses) via

$$(P,\,Q) = \Delta(P)Q\,\big|_{X=0}\,. \tag{5}$$

One can check that if $\{X_i\}_{i=1,2,\ldots,n=\dim\mathfrak{g}}$ is an orthonormal basis in $\mathfrak{g}$, then the monomials

$$\frac{X^k}{\sqrt{k!}} := \frac{X_1^{k_1}\cdots X_n^{k_n}}{\sqrt{(k_1)!\cdots(k_n)!}}$$

form an orthonormal basis in $S(\mathfrak{g})$.

The bilinear form (5) extends further to the space $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})\otimes S(\mathfrak{g})$ (which can be viewed as the space of matrix-valued polynomials on $\mathfrak{g}^*$). For decomposable elements this extension is given by

(5′)

$$(A_0\otimes P,\,B_0\otimes Q) := \mathrm{tr}(A_0B_0)(P,\,Q) \quad \text{for} \quad A_0, B_0 \in \mathrm{Mat}_{d(\lambda}(\mathbb{C}),\ \ P, Q \in S(\mathfrak{g}).$$

Note also that the extended form has the following useful property:

$$(AB,\,C) = (A,\,\Delta(B)C). \tag{6}$$

In other words, the matrix differential operator $\Delta(B)$ acting from the left is adjoint to the operator of right multiplication by B.

Lemma 1. *Let $P \in S(\mathfrak{g})$ be an invariant polynomial (i.e. we assume that P belongs to $I(\mathfrak{g}) = S(\mathfrak{g})^G$). Then the matrix*

$$M_P := \pi_\lambda(X_i)\otimes\partial^i P \tag{7}$$

belongs to $\mathcal{C}_\lambda(\mathfrak{g})$.

Proof. Direct verification of (2′). □

We call M_P an **M-type element** of $\mathcal{C}_\lambda(\mathfrak{g})$.

Note that for $P = C := \frac{1}{2}X_iX^i$ we get a special element

$$M := M_C = \pi_\lambda(X_i)\otimes X^i, \tag{8}$$

which belongs to $\mathcal{C}_\lambda(\mathfrak{g})$ (resp. to $\mathcal{Q}_\lambda(\mathfrak{g})$) if we interpret X^i as an element of $S(\mathfrak{g})$ (resp. of $U(\mathfrak{g})$).

The remarkable fact is

Theorem M. *The M-type elements belong to the center of $\mathcal{C}_\lambda(\mathfrak{g})$.*

Proof. For $A \in \mathcal{C}_\lambda(\mathfrak{g})$, $P \in I(\mathfrak{g})$ we have

$$[M_P, A] = [\pi_\lambda(X_i), A]\, \partial^i P \overset{(2'),(4)}{=} \{A, X_i\}\, \partial^i P = \{A, P\} = 0.$$

$\square$

Below we use this theorem to show that many important classical family algebras are commutative.

1.2. The structure of the family algebras. Let us look more attentively at the $\mathfrak{g}$-module structure of $\operatorname{End} V_\lambda \cong \mathrm{Mat}_{d(\lambda)}(\mathbb{C})$. It splits into irreducible components:

$$\mathrm{Mat}_{d(\lambda)}(\mathbb{C}) = \bigoplus_i W_i, \tag{9}$$

where W_i is a simple $\mathfrak{g}$-module with highest weight μ_i.

We shall call representations (π_{μ_i}, W_i) the **children** of π_λ (resp. the dominant weights μ_i will be called the children of λ).

In the case where all μ_i are different, we shall say that there are no **twins**.

Suppose that V_λ admits a G-invariant bilinear form. In other words, assume that π_λ belongs to orthogonal or symplectic type.

In the orthogonal case for an appropriate basis in V_λ all matrices $\pi_\lambda(X)$, $X \in \mathfrak{g}$, are skew-symmetric. Then $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})$ as a $\mathfrak{g}$-module splits into symmetric and antisymmetric parts. We call **boys** those children that belong to antisymmetric part, and **girls** those that are in the symmetric part.

In the symplectic case, $d(\lambda) = 2n$ is even and there is a $\pi_\lambda(G)$-invariant skew-symmetric form in V_λ. We can choose a basis such that this form is given by the matrix

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

In this basis all matrices $\pi_\lambda(X)$, $X \in \mathfrak{g}$, are J-symmetric, i.e. have the form JS, where S is symmetric.

The $\mathfrak{g}$-module $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})$ again splits into two parts: J-symmetric and J-antisymmetric. Here we also call girls those children that are in the J-symmetric part, and boys those in the J-antisymmetric part.

Note, that in both cases there is a distinguished girl—the trivial representation $(\pi_0, \mathbb{C}\cdot 1)$ or $(\pi_0, \mathbb{C}\cdot J)$.

We also observe that the adjoint representation Ad is a common child to all non-trivial representations π_λ; the corresponding subspace is spanned by the matrix elements of M (see (8) above). For representations of orthogonal type it is a boy, while for those of symplectic type it is a girl. We shall see below that in some cases there are twins of type Ad.

The algebra $\mathrm{Mat}_{d(\lambda)} \otimes S(\mathfrak{g}^*)$ of matrix differential operators on $\mathfrak{g}^*$ acts on the algebra $\mathrm{Mat}_{d(\lambda)} \otimes S(\mathfrak{g})$ of matrix polynomials on $\mathfrak{g}^*$ according to the rule

$$(A_0 \otimes D) \cdot (B_0 \otimes P) = A_0 B_0 \otimes D(P).$$

Since Δ is a G-equivariant map (it is an isomorphism of $\mathfrak{g}$-modules), we conclude that the subalgebra of G-invariant matrix operators

$$\mathcal{D}_\lambda(\mathfrak{g}) := \left(\mathrm{Mat}_{d(\lambda)} \otimes S(\mathfrak{g}^*)\right)^G$$

coincides with $(1 \otimes \Delta)\, \mathcal{C}_\lambda(\mathfrak{g})$.

The action of this subalgebra preserves the subalgebra $\mathcal{C}_\lambda(\mathfrak{g}) \subset \mathrm{Mat}_{d(\lambda)} \otimes S(\mathfrak{g})$. For our goals the most important examples of elements of $\mathcal{D}_\lambda(\mathfrak{g})$ are

$$D_P := \Delta(M_P) = \sum_i \pi_\lambda(X_i) \otimes \Delta(\partial^i P),$$

and in particular

$$D := \Delta(M) = \sum_i \pi_\lambda(X_i) \otimes \partial^i.$$

1.3. Generalized exponents. Here we recall some remarkable results mostly due to B. Kostant [K].

The first result describes the structure of $\mathfrak{g}$-module $S(\mathfrak{g})$. Let $I(g) = S(\mathfrak{g})^G$. We identify $I(g)$ with the algebra $P(\mathfrak{g}^*)^G$ of G-invariant polynomials on $\mathfrak{g}^*$.

Let $I_+(g)$ be the augmentation ideal in $I(\mathfrak{g})$ (i.e. the ideal of polynomials vanishing at the origin). Denote by $J(\mathfrak{g})$ the ideal in $S(\mathfrak{g})$ generated by $I_+(g)$. The space $H(\mathfrak{g})$ of **harmonic polynomials** on $\mathfrak{g}^*$ is defined as the orthogonal complement to $J(\mathfrak{g})$ in $S(\mathfrak{g})$.

According to (6), $H(\mathfrak{g})$ can also be defined as the space of solutions h to the system of equations

$$\Delta(P)h = 0, \qquad P \in I_+(g). \tag{10}$$

(Of course, it is enough to consider the generators of $I_+(\mathfrak{g})$ in the role of P.)

Theorem 1 (Kostant). *a) There is an isomorphism of graded $\mathfrak{g}$-modules:*

$$S(\mathfrak{g}) \cong I(\mathfrak{g}) \otimes H(\mathfrak{g}). \tag{11}$$

b) Each irreducible representation π_λ has finite multiplicity in $H(\mathfrak{g})$.

More precisely, if $s = m_\lambda(0)$ is the multiplicity of the zero weight in V_λ, then there exist numbers $e_1(\lambda), \dots, e_s(\lambda)$ (not necessarily distinct) such that π_λ occurs in the homogeneous components $H^{e_1(\lambda)}(\mathfrak{g}), \dots, H^{e_s(\lambda)}(\mathfrak{g})$.

The numbers $e_1(\lambda), \dots, e_s(\lambda)$ are called the **generalized exponents** related to the representation π_λ.

Since $H(\mathfrak{g})$ is a self-dual $\mathfrak{g}$-module, the generalized exponents are the same for λ and λ^*.

The **ordinary exponents** correspond to the adjoint representation for which $\lambda = \psi$, the highest root of $\mathfrak{g}$, and $s = l$, the rank of $\mathfrak{g}$.

The multiplicity $m_\lambda(\mu)$ of the weight μ in V_λ is often denoted by $K_{\lambda\mu}$ and called **Kostka number**. It has the remarkable q-analog: the so-called **Kostka polynomial** $K_{\lambda\mu}(q)$, which can be expressed in terms of the q-analog of the Kostant formula for ordinary multiplicities:

$$K_{\lambda\mu}(q) = \sum_{w \in W} (-1)^{l(w)} \mathcal{Q}(w(\lambda+\rho) - (\mu+\rho) \mid q), \tag{12}$$

where $\mathcal{Q}$ is the q-analog of the Kostant partition function and is defined by

$$\sum_{\nu \in Q_+} \mathcal{Q}(\nu \mid q) e^\nu = \prod_{\alpha > 0} (1 - qe^\alpha)^{-1}. \tag{13}$$

Theorem 2 (Hesselink). *The polynomials $K_{\lambda\mu}(q)$ for $\mu = 0$ coincide with generating functions for generalized exponents:*

$$K_{\lambda 0}(q) = \sum_{i=1}^{K_{\lambda 0}} q^{e_i(\lambda)}. \tag{14}$$

We observe that though the Hesselink result looks very natural and elegant, its practical use is rather restricted. The reason is that the explicit formula for generalized exponents which follows from Hesselink result is too involved. For Lie algebras of rank ≥ 3 it is practically uncomputable.

It is very interesting to compare this approach with the so-called Gelfand-Tsetlin patterns, which label a basis in V_λ. These patterns have the form

$$\begin{array}{ccccccccccc}
m_{1,1} & & m_{1,2} & & \dots & \dots & \dots & & m_{1,n-1} & & m_{1,n} \\
& m_{2,1} & & m_{2,2} & & \dots & & m_{2,n-2} & & m_{2,n-1} & \\
& & \dots & & & \dots & & & \dots & & \\
& & & m_{n-1,1} & & & & m_{n-1,2} & & & \\
& & & & & m_{n,1} & & & & &
\end{array}$$

with integers m_{ij} satisfying

$$m_{i,j} \geq m_{i+1,j} \geq m_{i,j+1}.$$

The weight of a vector v_M corresponding to a pattern M is equal to

$$m_{n,1},\ m_{n-1,1} + m_{n-1,2} - m_{n,1},\ \dots,\ \sum_j m_{1j} - \sum_j m_{2,j}.$$

One can show that each vector v_M of zero weight contributes to $K_{\lambda\mu}(q)$ a summand of the form $q^{c(M)}$, where $c(M)$ is a certain combinatorial function (the so-called **charge**) of the pattern M.

For the case $\mathfrak{g} = sl(4)$ a pattern of zero weight looks like

$$\begin{array}{ccccccccl}
m_1 & & m_2 & & m_3 & & m_4 & & (m_1+m_2+m_3+m_4=0) \\
& l_1 & & l_2 & & l_3 & & & (l_1+l_2+l_3=0) \\
& & p & & -p & & & & \\
& & & 0 & & & & &
\end{array}$$

I succeeded to compute the final expression only in several particular cases.

For example, for the pattern of the form

$$\begin{array}{ccccccc}
m & & 0 & & 0 & & -m \\
& l & & 0 & & -l & \\
& & p & & -p & & \\
& & & 0 & & &
\end{array}$$

we have $c(M) = m + l + p$, $K_{\lambda 0} = \frac{(m+1)(m+2)}{2} = \binom{m+2}{2}$, and $K_{\lambda 0}(q)$ is equal to the sum of all monomials from the following triangle table:

$$\begin{array}{ccccccccc} q^m & & q^{m+1} & & \dots & & q^{2m-1} & & q^{2m} \\ & q^{m+2} & & & \dots & & & q^{2m+1} & \\ & & \dots & & \dots & & \dots & & \\ & & & q^{3m-2} & & q^{3m-1} & & & \\ & & & & q^{3m} & & & & \end{array}$$

which leads to the expression

$$\begin{aligned} K_{\lambda 0}(q) = {} & q^m \begin{bmatrix} m+2 \\ 2 \end{bmatrix}_q = q^m \cdot \frac{q^{m+2}-1)(q^{m+1}-1)}{(q-1)(q^2-1)} \\ & = q^m + q^{m+1} + 2q^{m+2} + 2q^{m+3} + 3q^{m+4} + 3q^{m+5} + \cdots \\ & \quad + \left[\frac{m}{2}\right] q^{2m} + \cdots + 2q^{3m-3} + 2q^{3m-2} + q^{3m-1} + q^{3m}. \end{aligned}$$

Remark 1. The general formula for $c(M)$ in the case of $sl(4)$ has been recently found by my student R. Masenten. It looks like

$$c(M) = p + |l_2| + \max\left(m_1 + l_1 + l_2,\ -l_2 - l_3 - m_4,\ m_1 + m_2 + p\right). \tag{15}$$

It is a challenging problem to simplify, explain and generalize this cumbersome formula.

Remark 2. The ordinary multiplicity of the zero weight in a unirrep π_λ of the compact form $K \cong SU(n)$ of G can be written as follows. Let $M(K)$ be the algebra of (signed) measures on K with convolution product. Pick up a maximal torus $T \subset K$ and denote by δ_T the element of $M(K)$ given by the normalized Haar measure on T.

Then $\pi_\lambda(\delta_T)$ is the projector to V_λ^0 and

$$K_{\lambda 0}(q) = \operatorname{tr} \pi_\lambda(\delta_T).$$

Of course, we can replace the chosen torus T in this formula by any other (they are all conjugate in K). We can also take the average over all possible tori. Then we get a measure which is absolutely continuous with respect to Haar measure and has density

$$f(u) = \frac{n!}{\prod_{i \neq j} |\lambda_i - \lambda_j|},$$

where λ_i are eigenvalues of u.

It is natural to ask if there exists a simple graded version $f(u \,|\, q)$ of $f(g)$ such that

$$K_{\lambda 0}(q) = \operatorname{tr} \pi(f(u \,|\, q)).$$

For the case $K = SU(2)$ we can put

$$f(u \,|\, q) = \frac{1+q}{1 - q \cdot \operatorname{tr}(u^2) + q^2}. \tag{16}$$

1.4. Relation between family algebras and generalized exponents. Now let $\lambda \in P_+$ be a dominant weight such that π_λ has k children with no twins. We denote by $W_1, \ldots, W_k$ the corresponding irreducible subspaces in $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})$ and by $\mu_1, \ldots, \mu_k$ the highest weights of corresponding representations.

We denote by p_i the projection to W_i in $\mathrm{Mat}_{d(\lambda)}(\mathbb{C})$ and use the same notation for its extension to $\mathcal{C}_\lambda(\mathfrak{g}^*)$ (which is the restriction of $p_i \otimes 1$).

Then the subspace $\mathcal{C}_\lambda(\mathfrak{g}^*)_i := p_i(\mathcal{C}_\lambda(\mathfrak{g}^*)) = (W_i \otimes S(\mathfrak{g}))^G = (W_i \otimes I(\mathfrak{g}) \otimes H(\mathfrak{g}))^G$ is a free $I(\mathfrak{g})$-module of rank $K_{\mu_i 0}$. In fact, it has the grading inherited from $S(\mathfrak{g})$ and its Poincaré series is given by

$$ch_{\mathcal{C}_\lambda(\mathfrak{g}^*)_i}(q) = ch_{I(\mathfrak{g})}(q) \cdot ch_{H(\mathfrak{g})_{\mu_i^*}}(q) = \frac{K_{\mu_i 0}(q)}{\prod_{k=1}^{l}(1-q^{d_k})},$$

where d_k, $1 \leq k \leq l$, are the degrees of the generators of $I(\mathfrak{g})$. (It is well known that $d_k = e_k + 1$, where $e_k = e_k(\psi)$ are ordinary exponents.)

So, we get the following result:

Theorem 1. *The algebra $\mathcal{C}_\lambda(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module of rank $m = \sum_{i=1}^{k} m_{\mu_i}(0)$.*

Remark 3. Assume that all weights of the representation π_λ are simple (i.e. have multiplicity 1). Then the zero weight subspace in $\mathrm{Mat}_{d(\lambda)}(\mathbb{C}) \cong \mathrm{End}\, V_\lambda$ is the subspace of diagonal matrices so that $m = d(\lambda)$. I do not know the meaning of the corresponding q-analog.

Theorem 2. *The algebra $\mathcal{C}_\lambda(\mathfrak{g})$ is commutative if and only if all weights of the representation π_λ have multiplicity 1.*

It follows from the following general result:

Theorem 3. *Let $K(\mathfrak{g})$ be the field generated by the zero weight subalgebra $S(\mathfrak{g})^0$ of $S(\mathfrak{g})$, let μ_i be children of λ, and $\delta(\mu_i)$ their multiplicities. Then the algebra $\mathcal{K}_\lambda(\mathfrak{g}) := \mathcal{C}_\lambda(\mathfrak{g}) \otimes_{S(\mathfrak{g})^0} K(\mathfrak{g})$ is isomorphic to the direct sum of matrix algebras:*

$$\mathcal{K}_\lambda(\mathfrak{g}) \cong \bigoplus_i \mathrm{Mat}_{\delta(\mu_i)}(K(\mathfrak{g})). \tag{17}$$

Sketch of the proof. We shall regard elements of $\mathcal{K}_\lambda(\mathfrak{g})$ as rational matrix-valued functions on $\mathfrak{g}^*$ with values in $\mathrm{Mat}_\lambda(\mathbb{C})$. The condition $(2'')$ implies that this function is uniquely determined by its restriction to $\mathfrak{h}^* \subset \mathfrak{g}^*$.

Conversely, this restriction can be any rational function on $\mathfrak{h}^*$ with values in the zero weight subspace $\mathrm{Mat}_\lambda(\mathbb{C})^0$. (It follows from the birational isomorphism $G \cong G/B \times H$.) But the latter is exactly the direct sum $\bigoplus_i \mathrm{Mat}_{\delta(\mu_i)}(\mathbb{C})$. ☐

2. Family algebras for standard representations of classical simple Lie algebras

Here we illustrate the general theory described above by examples related to the simplest representations of classical simple Lie algebras. It can be viewed as a beautiful exercise in linear algebra which ties together many classical results.

One of interesting consequences of the results listed below is the fact that all family algebras in question are commutative. It is not true for the quantum algebras and for some classical algebras related to other representations.

2.1. The case $\mathfrak{g} = \mathbf{A_n} \cong \mathfrak{sl}(n+1,\,\mathbb{C})$. As usual it is slightly more convenient to consider $\mathfrak{g}$ together with the bigger algebra $\widetilde{\mathfrak{g}} = \mathfrak{gl}(n+1,\,\mathbb{C})$.

Denote by $\{E_{ij}\}_{1\le i,j\le n+1}$ the standard basis in $\widetilde{\mathfrak{g}}$, so that $\pi(E_{ij})$ is a matrix unit with the only non-zero entry in the i-th row and j-th column.

Let $\pi = \pi_{\omega_1}$ be the standard representation of $\widetilde{\mathfrak{g}}$ with the highest weight

$$\omega_1(E_{ii}) = \delta_{i,1}.$$

The irreducible subspaces of $\mathrm{Mat}_{n+1}(\mathbb{C})$ are W_0 and W_1, consisting respectively of scalar and traceless matrices.

It follows that π has two children: the trivial representation π_0 and the representation π_ψ with the highest weight

$$\psi(E_{ii}) = \delta_{i,1} - \delta_{i,n+1}.$$

Note, that the restriction of π_ψ on $\mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$.

The element $\widetilde{M} \in \mathcal{C}_\psi(\widetilde{\mathfrak{g}})$ has the form

$$\widetilde{M} = \sum_{i,j=0}^{n} \pi(E_{ij}) \otimes E_{ji} = \begin{pmatrix} E_{00} & \dots & E_{n0} \\ \dots & \dots & \dots \\ E_{0n} & \dots & E_{nn} \end{pmatrix}.$$

The element $M \in \mathcal{C}_\pi(\mathfrak{g})$ is just the traceless part of $\widetilde{M}$. For example, for $n = 2$ we have

$$M = \begin{pmatrix} \frac{1}{3}(2H_\alpha + H_\beta) & X_{-\alpha} & X_{-\gamma} \\ X_\alpha & \frac{1}{3}(H_\beta - H_\alpha) & X_{-\beta} \\ X_\gamma & X_\beta & -\frac{1}{3}(H_\alpha + 2H_\beta) \end{pmatrix}$$

where we use the standard notation: $X_\alpha = E_{01}$, $X_\beta = E_{12}$, $X_\gamma = E_{02}$, $X_{-\alpha} = E_{10}$, $X_{-\beta} = E_{10}$, $X_{-\gamma} = E_{20}$, $H_\alpha = E_{00} - E_{11}$, $H_\beta = E_{11} - E_{22}$.

Theorem A. *a) The algebra $I(\mathfrak{g})$ coincides with $\mathbb{C}\,[\mathrm{tr}\,(M^k),\ 2 \le k \le n+1]$ and $\mathcal{C}_\pi(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module of rank $n+1$ generated by elements M^k, $0 \le k \le n$.*

b) The algebra $I(\widetilde{\mathfrak{g}})$ coincides with $\mathbb{C}\,[\mathrm{tr}\,(\widetilde{M}^k)]$, $1 \le k \le n+1$, and $\mathcal{C}_\pi(\widetilde{\mathfrak{g}})$ is a free $I(\widetilde{\mathfrak{g}})$-module of rank $n+1$ generated by $\widetilde{M}^k$, $0 \le k \le n$.

Sketch of the proof. We start with part b). By the very definition $p_0(\mathcal{C}_\pi(\widetilde{\mathfrak{g}}))$ is isomorphic to $I(\widetilde{\mathfrak{g}})$: it consists of scalar matrices with elements from $I(\widetilde{\mathfrak{g}})$. So, the elements $I_k = \mathrm{tr}(\widetilde{M}^k)$ belong to $I(\widetilde{\mathfrak{g}})$. On the other hand, it is well known that the polynomials $I_1, \dots, I_{n+1}$ freely generate $I(\widetilde{\mathfrak{g}})$ (the main theorem on symmetric functions).

It is clear that for any $C \in I(\widetilde{\mathfrak{g}})$ the matrix $D(C)$ belongs to $\mathcal{C}_\pi(\widetilde{\mathfrak{g}})$. On the other hand, for any $A \in \mathcal{C}_\pi(\widetilde{\mathfrak{g}})$ the polynomial $\mathrm{tr}\,(AM)$ belongs to $I(\widetilde{\mathfrak{g}})$. The maps $\alpha : C \mapsto D(C)$ and $\beta : A \mapsto \mathrm{tr}\,(AM)$ are almost reciprocal: $\beta \circ \alpha = \deg C$ for homogeneous C, hence both α and β are invertible. This proves b).

Part a) easily follows from b). $\square$

Corollary. *The ordinary exponents of $\mathfrak{g}$ are $1, 2, \dots, n$.*

2.2. The case $\mathfrak{g} = \mathbf{B_n} \cong \mathfrak{so}(2n+1,\,\mathbb{C})$. Here the basic representation $\pi = \pi_{\omega_1}$ has dimension $d(\omega_1) = 2n+1$ and is orthogonal. So the space $\mathrm{Mat}_{2n+1}(\mathbb{C})$ as a $\mathfrak{g}$-module splits into symmetric and antisymmetric parts. Actually, in this case there are two girls and one boy.

Namely, the symmetric part has dimension $(2n+1)(n+1)$ and contains the trivial subrepresentation W_0 acting on the space of scalar matrices. The complementary subspace W_2 of traceless symmetric matrices is irreducible and has the highest weight $2\omega_1$.

The antisymmetric part W_1 is irreducible and isomorphic to the adjoint representation with highest weight ω_2.

The matrix M is antisymmetric and its k-th power is symmetric for even k and antisymmetric for odd k.

Theorem B. *a) The algebra $I(\mathfrak{g})$ coincides with $\mathbb{C}\ [\mathrm{tr}\,(M^{2k}),\ 1 \le k \le n]$ and $\mathcal{C}_\pi(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module of rank $2n+1$ generated by M^k, $0 \le k \le 2n$.*

b) The $I(\mathfrak{g})$-module $p_2(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by $p_2(M^{2k})$, $1 \le k \le n$, while the $I(\mathfrak{g})$-module $p_1(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by M^{2k-1}, $1 \le k \le n$.

We omit the proofs of this and the next theorem because they follow the same scheme as above.

Corollary. *The exponents of $\pi_{2\omega_1}$ are $2, 4, \dots, 2n$ and the exponents of π_{ω_2} (the ordinary exponents) are $1, 3, \dots, 2n-1$.*

2.3. The case $\mathfrak{g} = \mathbf{C_n} \cong \mathfrak{sp}(2n, \mathbb{C})$. This time the basic representation $\pi = \pi_{\omega_1}$ has dimension $2n$ and is symplectic. The space $\mathrm{Mat}_{2n}(\mathbb{C})$ splits into 1-dimensional space $W_0 = \mathbb{C} \cdot J$, the subspace W_2 of J-symmetric matrices orthogonal to J, and the subspace W_1 of J-antisymmetric matrices.

Thus, there are two girls: 0 and $2\omega_1$, and one boy: ω_2.

The matrix M is J-symmetric and its k-th power is J-symmetric for odd k and J-antisymmetric for even k.

Theorem C. *a) The algebra $I(\mathfrak{g})$ coincides with $\mathbb{C}\ [\mathrm{tr}\,(M^{2k}),\ 1 \le k \le n]$, and $\mathcal{C}_\pi(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module of rank $2n$ generated by M^k, $0 \le k \le 2n-1$.*

b) The $I(\mathfrak{g})$-module $p_2(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by $p_2(M^{2k})$, $1 \le k \le n-1$, while the $I(\mathfrak{g})$-module $p_1(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by M^{2k-1}, $1 \le k \le n$.

Corollary. *The exponents of π_{ω_2} are $2, 4, \dots, 2n-2$, and the exponents of $\pi_{2\omega_1}$ (the ordinary exponents) are $1, 3, \dots, 2n-1$.*

2.4. The case $\mathfrak{g} = \mathbf{D_n} \cong \mathfrak{so}(2n, \mathbb{C})$. The basic representation $\pi = \pi_{\omega_1}$ of $\mathfrak{g}$ has dimension $d(\omega_1) = 2n$. Here again, as in the case $\mathbf{B}_n$, there are two girls: 0 and $2\omega_1$, and one boy: ω_2.

More precisely, the symmetric part has dimension $n(2n+1)$ and contains the trivial subrepresentation W_0 of scalar matrices and the complementary irreducible subspace W_2 of traceless symmetric matrices with highest weight $2\omega_1$.

The antisymmetric part W_1 is irreducible and isomorphic to the adjoint representation with highest weight ω_2.

The matrix M is antisymmetric and its k-th power is symmetric for even k and antisymmetric for odd k.

Theorem D. *a) The algebra $I(\mathfrak{g})$ coincides with the polynomial algebra generated by $\mathrm{tr}\,(M^{2k})$, $1 \le k \le n-1$, and $Pf(M)$. The algebra $\mathcal{C}_\pi(\mathfrak{g})$ is a free $I(\mathfrak{g})$-module of rank $2n$ generated by elements M^k, $0 \le k \le 2n-1$, and $M^{-1} \cdot Pf(M)$.*

b) The $I(\mathfrak{g})$-module $p_2(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by $p_2(M^{2k})$, $1 \le k \le n-1$, while the $I(\mathfrak{g})$-module $p_1(\mathcal{C}_\pi(\mathfrak{g}))$ is spanned by M^{2k-1}, $1 \le k \le n-1$, and $M^{-1} \cdot Pf(M)$.

Here the scheme of the proof is essentially the same with one exception. There exists an element in $I(\mathfrak{g})$ which cannot be expressed in terms of $\operatorname{tr} M^k$, $k \in \mathbb{N}$. It is the so-called **Pfaffian** $\operatorname{Pf}(M) := \sqrt{\det M}$.

Accordingly, the space $p_1(\mathcal{C}_\pi(\mathfrak{g}))$ contains, besides all odd powers of M, an additional element $M^{-1} \cdot \operatorname{Pf}(M)$. It is a nice exercise in matrix algebra to prove that the entries of this matrix are indeed polynomials in matrix elements of M.

Corollary. *The exponents of* $\pi_{2\omega_1}$ *are* $2, 4, \ldots, 2n-2$, *and the exponents of* π_{ω_2} *(the ordinary exponents) are* $1, 3, \ldots, 2n-3$ *and* $n-1$.

3. Other examples

3.1. The case of a 7-dimensional representation of $\mathbf{G_2}$. In this section I use some computations performed by my student N. Rojkovskaya.

The simplest exceptional Lie algebra of type $\mathbf{G_2}$ admits a $\mathbb{Z}_3$-grading such that

$$\mathfrak{g}_0 \cong \mathfrak{sl}(3, \mathbb{C}), \qquad \mathfrak{g}_1 \cong V, \quad \mathfrak{g}_2 \cong V^*,$$

where V and V^* are dual fundamental modules for $\mathfrak{sl}(3, \mathbb{C})$. We realize $X \in \mathfrak{g}_0$ as a traceless complex 3×3 matrix, $v \in V$ as a column 3-vector with coordinates v^1, v^2, v^3, and $f \in V^*$ as a row 3-vector with coordinates f_1, f_2, f_3.

The 7-dimensional representation of $\mathbf{G_2}$ has the form:

$$\mathcal{X} = \begin{pmatrix} X & v & A(f) \\ f & 0 & -\check{v} \\ B(v) & -\check{f} & -\check{X} \end{pmatrix}. \tag{18}$$

Here the check symbol denotes the transposition with respect to the second diagonal, and the following notation is used:

$$A(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} -f_2 & f_3 & 0 \\ f_1 & 0 & -f_3 \\ 0 & -f_1 & f_2 \end{pmatrix}, \quad B(v) = -A(\check{v}) = \frac{1}{\sqrt{2}} \begin{pmatrix} v^2 & -v^1 & 0 \\ -v^3 & 0 & v^1 \\ 0 & v^3 & -v^2 \end{pmatrix}.$$

The fact that matrices (17) form a Lie algebra is equivalent to the following simple statement.

Lemma 2. *If* $X \in \mathfrak{sl}(3, \mathbb{C})$, $v \in V$, $f \in V^*$, *then*
a) $A(-fX) = XA(f) + A(f)\check{X}$; *b)* $B(Xv) = -\check{X}B(v) - B(v)X$;
c) $A(f)B(\check{v}) = \frac{1}{2}(vf - fv \cdot 1)$.

We see that the 7-dimensional representation $\pi = \pi_{\omega_2}$ has 4 children: two girls and two boys.

The girls are the trivial representation and the 27-dimensional representation with the highest weight $2\omega_2$.

The boys are 14-dimensional and 7-dimensional representations with the highest weights ω_1 and ω_2 respectively.

We denote by p_1, p_{27}, p_{14} and p_7 the corresponding projections.

Theorem G. *a) The algebra* $I(\mathfrak{g})$ *is generated by* $\operatorname{tr} M^k$, $k = 2, 6$.
b) The $I(\mathfrak{g})$*-module* $p_{27}(\mathcal{C}_\pi(\mathfrak{g}))$ *is spanned by* $p_{27}(M^{2k})$, $1 \le k \le 3$,
the $I(\mathfrak{g})$*-module* $p_{14}(\mathcal{C}_\pi(\mathfrak{g}))$ *is spanned by* $p_{14}(M^k)$, $k = 1, 5$,
the $I(\mathfrak{g})$*-module* $p_7(\mathcal{C}_\pi(\mathfrak{g}))$ *is spanned by* $p_7(M^3)$.

Corollary. *The generalized exponents for $2\omega_2$ are $2, 4, 6$; the unique generalized exponent for ω_2 is 3 and the ordinary exponents are $1, 5$.*

3.2. The case of the adjoint representation of $\mathbf{A_n}$. The adjoint representation $\pi = \pi_{\omega_1+\omega_n}$ is orthogonal. Therefore, the space $\mathrm{Mat}_{n^2+2n}(\mathbb{C})$ splits as a $\mathfrak{g}$-module into symmetric and antisymmetric parts.

More detailed analysis (see e.g. [OV], Table 5) shows, that there are 7 children (4 girls and 3 boys). In the table we show their dimensions and the numbers of exponents.

Highest weight	boy/girl	dimension	number of exponents
$2\omega_1 + 2\omega_n$	girl	$\frac{n(n+1)^2(n+4)}{4}$	$\frac{n(n+1)}{2}$
$2\omega_1 + \omega_{n-1}$	boy	$\frac{(n-1)n(n+2)(n+3)}{4}$	$\frac{n(n-1)}{2}$
$\omega_2 + 2\omega_n$	boy	$\frac{(n-1)n(n+2)(n+3)}{4}$	$\frac{(n+1)(n-2)}{2}$
$\omega_2 + \omega_{n-1}$	girl	$\frac{(n+1)^2(n^2-4)}{4}$	$\frac{n(n-1)}{2}$
$\omega_1 + \omega_n$	girl	$n(n+2)$	n
$\omega_1 + \omega_n$	boy	$n(n+2)$	n
0	girl	1	1

(For small n some degenerations occur: for $n = 2$ there are 3 girls and 3 boys and for $n = 1$ there are two girls and one boy.)

There are two special elements in our algebra related to two covariant pairings $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. The first is antisymmetric and given by the commutator, while the second is symmetric and given by the traceless part of the anticommutator. Below we give the precise formula for these elements.

We introduce the notation $N = [\frac{\dim \mathfrak{g}}{2}] = [\frac{n^2}{2}] + n$ and enumerate the basis vectors in $\mathfrak{g}$ by integers from $-N$ to N (including zero if n is even), so that the Ad-invariant form on $\mathfrak{g}$ look like

$$(X_i, X_j) = \begin{cases} 1 & \text{if} \quad i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $n = 2$ we can put

$$X_{\pm 1} = X_{\pm\alpha}, \quad X_{\pm 2} = X_{\pm\beta}, \quad X_{\pm 3} = X_{\pm\gamma}, \quad X_{\pm 4} = H_{\pm},$$

where α and β are simple roots, $\gamma = \alpha + \beta$ and

$$H_+ = \frac{\epsilon H_\alpha - \epsilon^2 H_\beta}{\sqrt{3}}, \qquad H_- = \frac{\epsilon^2 H_\alpha - \epsilon H_\beta}{\sqrt{3}}, \qquad \epsilon = e^{2\pi i/3}.$$

Then we define matrices A and S in the family algebra $\mathcal{A} := \mathcal{C}_{\omega_1+\omega_n}(\mathbf{A_n})$ by

$$A^j_k = [X_{-j}, X_k], \qquad S^j_k = \{X_{-j}, X_k\}_0. \tag{19}$$

Note that A is antisymmetric with respect to the second diagonal, while S is symmetric. Indeed, by the very definition we have

$$A^{-k}_{-j} = -A^j_k, \qquad S^{-k}_{-j} = S^j_k.$$

In fact, A coincides with the element M introduced above. Hence, it commutes with S.

This algebra is non-commutative for $n \geq 2$. The further study of it is very interesting.

3.3. The quantum family algebras for $\mathbf{A_1}$. Let α be the simple root of $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{C})$, and $\omega = \frac{1}{2}\alpha$ the fundamental weight. We choose the standard basis $X_1 = E$, $X_0 = \frac{1}{2}H$, $X_{-1} = F$. Denote by C the central element $X_0^2 + \frac{1}{2}(X_1X_{-1} + X_{-1}X_1)$. The algebra $\mathcal{Q}_n := \mathcal{Q}_{n\omega}(\mathfrak{g})$ contains the element

$$A_n = \begin{pmatrix} \frac{1}{2}nH & \sqrt{1\cdot n}F & 0 & 0 & \dots & 0 & 0 \\ \sqrt{1\cdot n}E & \frac{1}{2}(n-2)H & \sqrt{2\cdot(n-1)}F & 0 & \dots & 0 & 0 \\ 0 & \sqrt{2\cdot(n-1)}E & \frac{1}{2}(n-4)H & \sqrt{3\cdot(n-2)}F & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2}(2-n)H & \sqrt{n\cdot 1}F \\ 0 & 0 & 0 & 0 & \dots & \sqrt{n\cdot 1}E & -\frac{1}{2}nH \end{pmatrix}.$$

For $n = 1$ we have $A_1 = \begin{pmatrix} \frac{1}{2}H & F \\ E & -\frac{1}{2}H \end{pmatrix}$. In this case it is not difficult to check directly that $\mathcal{Q}_1$ is a free $\mathbb{C}[C]$-module with generators 1 and A. The same is true in general:

Theorem $\mathcal{Q}$. *The quantum family algebra $\mathcal{Q}_n$ is a free $\mathbb{C}[C]$-module with $n+1$ generators $1, A, \dots, A^n$. In particular, the algebra is commutative and eventually isomorphic to the corresponding classical family algebra.*

Note, however, that the defining relations for $\mathcal{Q}_n$ and $\mathcal{C}_n$ are different. For the latter algebra it is the usual Cayley identity,

$$A^{n+1} + \sum_{k=1}^{n+1} c_k A^{n+1-k} = 0, \qquad \text{where} \quad c_k = (-1)^k \operatorname{tr}(\wedge^k A).$$

The corresponding quantum identity for $n = 2$ looks like $A^2 - A + c_2 \cdot 1 = 0$, while the Cayley identity is $A^2 + c_2 \cdot 1 = 0$.

4. Some open questions

We list here some cases where the family algebras seemed to admit the full description. (The order reflects the anticipated difficulty of the problem.)

1. The classical algebras for adjoint representations of classical groups.
2. The quantum algebras for the standard representations of classical groups.
3. The quantum algebra for the 7-dimensional representations of $\mathbf{G_2}$.
4. The classical algebra for the minimal representation of the exceptional group $\mathbf{F_4}$.
5. The same for minimal representations of the exceptional groups $\mathbf{E_6}$, $\mathbf{E_7}$.
6. The classical algebra for the adjoint representation of $\mathbf{E_8}$.
7. In general, it would be very interesting to find out which quantum family algebras are commutative and which classical algebras are spanned over $I(\mathfrak{g})$ by powers of M or analogous elements related to other generators of $I(\mathfrak{g})$.

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Department of Mathematics, The University of Pennsylvania, Philadelphia, PA 19104
E-mail address: kirillov@math.upenn.edu