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PROOF OF THE DOUBLE BUBBLE CONJECTURE

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ABSTRACT. We prove that the standard double bubble provides the least-area way to enclose and separate two regions of prescribed volume in $\mathbb{R}^3$.

1. History

Archimedes and Zenodorus (see [K, p. 273]) claimed and Schwarz [S] proved that the round sphere is the least-perimeter way to enclose a given volume in $\mathbb{R}^3$. The Double Bubble Conjecture, long assumed true ([P, pp. 300–301], [B, p. 120]) but only recently stated as a conjecture [F1, Section 3], says that the familiar double soap bubble in Figure 1, consisting of two spherical caps separated by a spherical cap (or flat disc), meeting at 120-degree angles, provides the least-perimeter way to enclose and separate two given volumes. The analogous result in $\mathbb{R}^2$ was proved by the 1990 Williams College "SMALL" undergraduate research Geometry Group [F2]. In 1995, Hass, Hutchings, and Schlafly [HHS] announced a computer-assisted proof for the case of equal volumes in $\mathbb{R}^3$. (See [M1], [HS1], [HS2], [Hu], [M2, Chapter 13].) Here we announce a proof [HMRR] of the general Double Bubble Conjecture, using stability arguments.

Theorem 1.1. *In $\mathbb{R}^3$, the unique perimeter-minimizing double bubble enclosing and separating regions R_1 and R_2 of prescribed volumes v_1 and v_2 is a standard double bubble as in Figure 1, consisting of three spherical caps meeting along a common circle at 120-degree angles. (For equal volumes, the middle cap is a flat disc.)*

Reichardt et al. [RHLS] have generalized our results to $\mathbb{R}^4$ and certain higher dimensional cases (when at least one region is known to be connected). The 2000 edition of [M2] treats bubble clusters through these current results.

2. Previous results

(See [M2, Chapters 13 and 14].) F. Almgren [A, Thm. VI.2] proved the existence and almost everywhere regularity of perimeter-minimizing bubble clusters enclosing k prescribed volumes in $\mathbb{R}^n$, using geometric measure theory. Taylor [T] proved that minimizers in $\mathbb{R}^3$ consist of smooth constant-mean-curvature surfaces meeting in threes at 120-degree angles along curves, which in turn meet in fours at isolated points. An idea suggested by White, written up by Foisy [F1, Thm. 3.4] and

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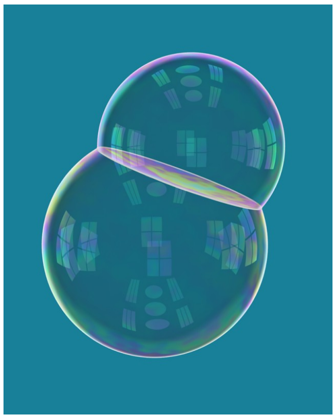

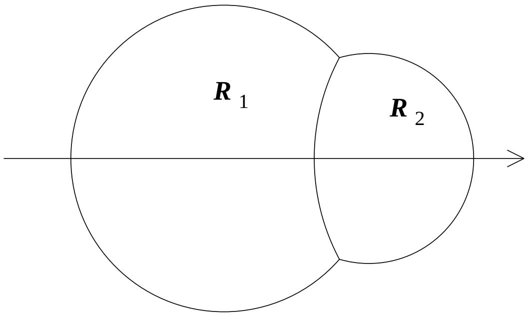

FIGURE 1. The standard double bubble provides the least-perimeter way to enclose and separate two prescribed volumes. [Computer graphics copyright John M. Sullivan, University of Illinois, *http://www.math.uiuc.edu/~jms/Images*.]

Hutchings [Hu, Thm. 2.6], shows that any perimeter-minimizing double bubble in $\mathbb{R}^n$ has rotational symmetry about some line. A major complication is that the regions are not *a priori* known to be connected. (If one tries to require the regions to be connected, they might in principle disconnect in the minimizing limit as thin connecting tubes shrink away.) Hutchings [Hu] developed new concavity and decomposition arguments to rule out "empty chambers" (bounded components of the exterior) and to bound the number of connected components of the two regions of a minimizer. In particular, a nonstandard minimizer in $\mathbb{R}^n$ consists of a central bubble with nested toroidal bands. For equal volumes in $\mathbb{R}^2$, there is only one band and at most a two-parameter family of possibilities. Hass and Schlafly [HS2] carried out a rigorous computer search of this family to eliminate these possibilities and prove the Double Bubble Conjecture for equal volumes. Earlier computer experiments of Hutchings and Sullivan had suggested that no such double bubbles were stable.

3. THE PROOF

As indicated above, a perimeter-minimizing double bubble is known to exist, to have rotational symmetry, and to consist of constant-mean-curvature surfaces of revolution ("surfaces of Delaunay"), meeting in threes at 120-degree angles along circles of revolution. In $\mathbb{R}^3$ the convexity and decomposition arguments of Hutchings [Hu, Thm. 4.2] imply easily that the larger region is connected and with careful computation that the smaller region has at most two components, as in Figure 2. Actually we give a less computational, stability argument to show that the smaller region has at most two components.

To rule out nonstandard minimizers, we use the following stability argument. Suppose that there is a nonstandard minimizer, and consider rotations about an axis A orthogonal to the axis of symmetry as in Figure 3. This axis can be chosen so that the places where the rotation vector field v is tangent to the double bubble separate the bubble into (at least) four pieces. Some linear combinations of the

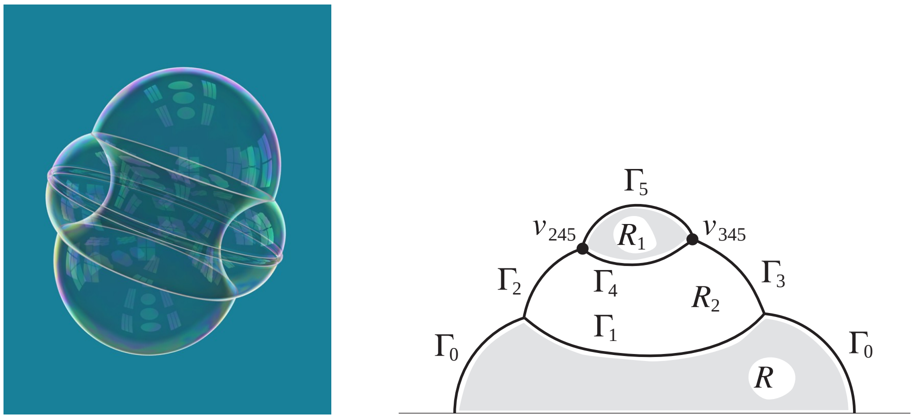

FIGURE 2. In this nonstandard double bubble, the smaller region has two components—a central bubble and a thin toroidal bubble around the outside—while the larger region is another toroidal bubble in between. Note that although this example consists of constant mean curvature surfaces meeting at 120-degree angles, it is not in equilibrium, because the two components of the disconnected region have unequal pressures. [Computer graphics copyright John M. Sullivan, University of Illinois, *http://www.math.uiuc.edu/ ~jms/Images*.]

restrictions of v to the four pieces vanish on one piece and respect the two volume constraints. By regularity for eigenfunctions, v must vanish on certain associated parts of the bubble, which therefore must be spherical or flat. When three surfaces meet and two are spherical or flat, so is the third. Lots of spherical or flat surfaces lead easily to a contradiction.

This argument is inspired by Courant's Nodal Domain Theorem [CH, p. 452], which says for example that the first eigenfunction is nonvanishing. Other applications of this principle to isoperimetric problems and to the study of volume-preserving stability have been given by Ritoré and Ros [RR], by Ros and Vergasta [RV], by Ros and Souam [RS], and by Pedrosa and Ritoré [PR]. Perhaps the simplest such phenomenon without constraints is that a circle of longitude on the unit sphere is unstable because the rotation vector field vanishes at the poles (conjugate points): rotating just half of it can be smoothed to reduce length.

Finding the required axis A requires consideration of a number of cases, as shown in Figure 3.

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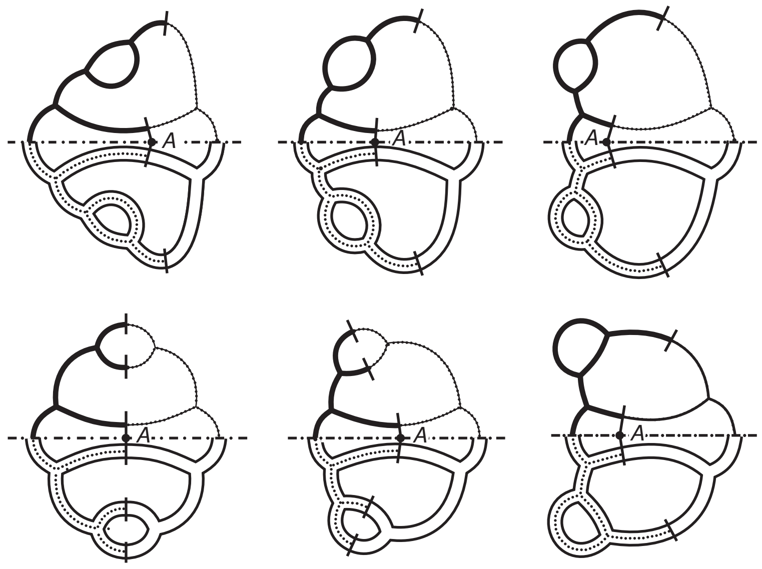

FIGURE 3. Case by case, one finds an "axis of instability" A perpendicular to the x-axis of symmetry. The places " | " where the rotational vector field is tangent to the surface divide the bubble into four pieces. [Drawing by James F. Bredt, copyright 2000 Frank Morgan.]

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