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SOLITONS ON PSEUDO-RIEMANNIAN MANIFOLDS: STABILITY AND MOTION

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ABSTRACT. This is an announcement of results concerning a class of solitary wave solutions to semilinear wave equations. The solitary waves studied are solutions of the form $\phi(t,x) = e^{i\omega t} f_\omega(x)$ to semilinear wave equations such as $\Box\phi + m^2\phi = \beta(|\phi|)\phi$ on $\mathbb{R}^{1+n}$ and are called nontopological solitons. The first preprint provides a new modulational approach to proving the stability of nontopological solitons. This technique, which makes strong use of the inherent symplectic structure, provides explicit information on the time evolution of the various parameters of the soliton. In the second preprint a pseudo-Riemannian structure $\underline{g}$ is introduced onto $\mathbb{R}^{1+n}$ and the corresponding wave equation is studied. It is shown that under the rescaling $\underline{g} \to \epsilon^{-2}\underline{g}$, with $\epsilon \to 0$, it is possible to construct solutions representing nontopological solitons concentrated along a time-like geodesic.

1. INTRODUCTION

Solitons and classical field theory. The subject of classical field theory involves the study of partial differential equations which describe the evolution and interaction of a collection of fields, i.e. functions on the space-time manifold $\mathbb{M}$ or other geometric objects such as sections, connections and metrics defined on vector bundles over $\mathbb{M}$. The gravitational field itself is described by a pseudo-Riemannian metric $\underline{g}$ on $\mathbb{M}$. The situation of a flat metric corresponds to the absence of a gravitational field, in which case one is led to the study of equations on the Minkowski space-time $(\mathbb{M}, \underline{g}) = (\mathbb{R}^{1+3}, \eta)$ where η is the metric

$$\eta = -dt^2 + \sum_{i=1}^{3} (dx^i)^2$$

in standard coordinates $(x^0 = t, x^1, x^2, x^3)$. The requirement of invariance under the Lorentz group, i.e. the set of (pseudo-orthogonal) linear transformations which preserve η, leads to PDE's of hyperbolic type such as the wave equation

$$\Box\phi \equiv \partial_t^2\phi - \Delta\phi = 0.$$

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This equation is an appropriate prototype for understanding essentially linear effects such as radiation. On the other hand important systems such as Einstein's equation and the Yang-Mills-Higgs equations are nonlinear and lead to different phenomena such as singularity formation (e.g. black holes) and energy concentration or localisation (e.g. monopoles and other solitons). Following [6] a soliton in classical field theory is, for present purposes, defined to be a spatially localised solution to the equations which does not decay or disperse as time progresses; the term is not intended to imply any properties of complete integrability of the PDE. As in [6] it is helpful to distinguish between *topological* solitons (such as BPS monopoles and Ginzburg-Landau vortices) which are stabilised by a nonzero winding number and *nontopological solitons* in which the stabilising mechanism is dynamical.[1] It is this latter case which is considered here: to be precise, nontopological solitons on $\mathbb{M}$ are here defined as solutions to the equation for $\phi : \mathbb{M} \to \mathbb{C}$,

$$\Box\phi = \mathcal{F}(\phi), \tag{1}$$

of the form

$$\phi(t,x) = e^{i\omega t} f_\omega(x). \tag{2}$$

The basic properties of these solutions are explained below in §2.

So far the discussion has been limited to the case when there is no gravitational field present and the metric is flat. In the presence of a gravitational field it is necessary to study the corresponding PDE's with a background metric $\underline{g}$ which itself evolves according to Einstein's equation. As an intermediate problem I consider here the case of a fixed background metric: mathematically this means the study of the semilinear wave equation on a pseudo-Riemannian manifold $(\mathbb{M}, \underline{g})$.

The geodesic hypothesis. In general relativity it is a standard assumption ([17]) that a "test-particle" introduced into the space-time will follow a geodesic with respect to the background metric (to highest order): this will be referred to as the *geodesic hypothesis*. In this work it is my aim to give a rigorous mathematical derivation of this hypothesis for the case of a fixed background metric. Of course there are many possible ways to formulate a corresponding precise mathematical question as it is necessary to translate the notion of "test-particle" into definite mathematical terms. Here I do this by taking particle to mean nontopological soliton; this is reasonable in view of the strong spatial localisation which these solitons exhibit[2]. In order to consider a *test*-particle it is necessary to also consider a rescaling of the metric

$$\underline{g} \to \epsilon^{-2}\underline{g} \qquad \text{with} \qquad \epsilon << 1$$

to ensure that the size of the particle is small compared to scales over which the metric varies. All in all this leads to the problem of constructing solutions to the

[1] Nontopological solitons correspond to what are called relative equilibria in Hamiltonian mechanics; see e.g. [7, Chapter 4].

[2] The choice of nontopological solitons to model test particles is not intended to be physically realistic, but rather to provide an interesting and relatively clean analytical problem in which to develop appropriate techniques.

equation[3]

$$\Box_{\underline{g}}\phi = \frac{1}{\epsilon^2}\mathcal{F}(\phi) \tag{3}$$

which are close to a nontopological soliton concentrated along a geodesic. This problem is given a solution in Theorem 7. The proof is an outgrowth of a new explicit approach to the stability of the solitary wave solutions on flat Minkowski space, as detailed in §3. Before explaining these results in detail a review of the basic properties of nontopological solitons will be given in the next section.

2. Nontopological solitons

In this section a general discussion of known existence and stability results for the class of solitons under investigation will be given. The domain will be $\mathbb{M} = \mathbb{R}^{1+n}$ with Minkowski metric $\eta = -dt^2 + \sum_{i=1}^n (dx^i)^2$ in standard coordinates.

Existence. Consider the semilinear wave equation for $\phi : \mathbb{M} \to \mathbb{C}$ with the following type of nonlinearity:

$$\partial_t^2\phi - \Delta\phi + m^2\phi = \beta(|\phi|)\phi, \quad \text{where } \beta : \mathbb{R} \to \mathbb{R} \text{ and } m^2 > 0. \tag{4}$$

Observe that this form of nonlinearity ensures that the equation is invariant under the action of S^1 by phase rotation $\phi \to \phi e^{i\chi}$, for any $\chi \in \mathbb{R}$. Nontopological solitons are solutions to (4) of the form

$$\phi(t,x) = e^{i\omega t} f_\omega(x), \tag{5}$$

where (for appropriate β) the function $f_\omega(x)$ is the unique positive radial solution on $\mathbb{R}^n$ of the *elliptic* equation

$$-\Delta f_\omega + (m^2 - \omega^2) f_\omega = \beta(f_\omega) f_\omega \qquad \text{with } \omega^2 < m^2. \tag{6}$$

Notice that although the solution is not independent of time it has a very simple time dependence: it maps out an orbit of the symmetry group S^1 as a function of t. This concept can of course be generalised to solutions which map out the orbit of a one-parameter subgroup of the symmetry group; see (8). Solutions of this form have been much studied in Hamiltonian mechanics and are called relative equilibria in that context ([7]); many examples of this type of solution for PDE's are given in [12, Chapter 7] and [4, 5].

Existence and uniqueness of $f_\omega > 0$ verifying (6) has been proved under very general conditions on β (see [3, 2, 8, 11, 12] and references therein). A useful model case to keep in mind is the *pure power nonlinearity*

$$\beta(|\phi|)\phi = |\phi|^{p-1}\phi,$$

in which case a unique positive radial solution to (6) exists if $1 < p < 1 + \frac{4}{n-2}$. For clarity the discussion is presented in this announcement mostly in the context of the pure power case, but very general conditions which are sufficient for the present development are made precise in [13, 14].

[3]Here $\Box_{\underline{g}}$ is the wave operator on a pseudo-Riemannian manifold, an explicit formula for which is given in (36) below.

Symmetry. The Minkowski metric is invariant under the Poincaré group, i.e. the group generated by translations $\mathbb{R}^n$ and the Lorentz group $SO(1,n)$ of linear transformations preserving the inner product $-dt^2 + \sum_{i=1}^n (dx^i)^2$. Thus together with the symmetry group S^1 referred to above there is a symmetry group

$$G = S^1 \times \text{Poincaré} \tag{7}$$

which acts on the space of solutions to (4). The spatial rotations of $\mathbb{R}^n$ leave (5) fixed since f_ω is radial. Since the space-time Lorentz rotations depend on a vector $u \in \mathbb{R}^n$ with $|u| < 1$, this gives altogether a $2n+1$-dimensional orbit through the solution (5). Since (5) depends also on the real parameter $\omega \in (-m, m)$, this provides a $2n+2$-dimensional family of solutions $\mathcal{S}$ in total.

Consider in particular the effect of a Lorentz transformation which takes the line $x = 0$ through the origin into the line $x = ut$. The action of this transformation on (5) provides a solution of (4) in which the soliton moves along the straight line $x = ut$; see (12)–(19) for explicit formulae. A question which is now very natural to raise in relation to the geodesic hypothesis is this: are there solutions to the equation on a manifold (3) which correspond to these? Theorem 7 states that in the limit $\epsilon \to 0$ there are indeed solutions in which the straight lines are replaced by time-like geodesics with respect to the metric $\underline{g}$. The proof of Theorem 7 is a development of a technique, outlined in the next section, which was developed in [13] to prove *modulational stability* of (5). In the remainder of this section I discuss some previous results on the stability problem.

Stability in the presence of symmetry. The stability of nontopological solitons (5) was first considered in [10] under the assumption of radial symmetry. A general framework for stability of solitary waves in the presence of symmetry in (infinite-dimensional) Hamiltonian systems was developed in [4, 5].

A crucial general point ([1, 12]) is that the appropriate notion of stability must take into account the symmetry group. So assume there is given a Hamiltonian system with a phase space X on which the group G acts

$$G \times X \to X,$$
$$(g, \phi) \mapsto g \cdot \phi$$

as a group of symmetries. A solitary wave (or relative equilibrium) solution is one which is the orbit of a one-parameter subgroup of G, i.e. there exists a one-parameter subgroup g_t and a point φ_0 such that

$$\phi_0(t) = g_t \cdot \varphi_0 \tag{8}$$

is an integral curve of the given Hamiltonian system. In the presence of a symmetry group G the appropriate notion of stability is this:

> *The solitary wave ϕ_0 is K-stable in the metric d, where K is a subgroup of G, if for Cauchy data close to $\phi_0(0)$ in metric d there is a global solution which remains uniformly close in metric d to the orbit $K \cdot \phi_0$ for all time.*

This definition is essentially a rewording of the usual definition of uniform stability with "K-orbit of solution" in place of "solution". The aim of a stability investigation would then be to find a subgroup K for which this holds ([5]).

To illustrate this point consider, as in [10], the stability of (5) with the assumption of radial symmetry. This assumption leaves a residual symmetry group S^1

acting on the system. In [10] it was proved that (5) is stable with respect to radially symmetric perturbations of the initial data for ω lying in the interval $\mathbf{I}$ which is defined by:

$$\mathbf{I} \equiv \left\{ \omega \in \mathbb{R} : \frac{\partial}{\partial \omega} \left(-\omega \|f_\omega\|_{L^2}^2 \right) > 0 \right\}. \tag{9}$$

This is called the *stability interval.* To be precise consider the Cauchy problem for (4) with radial initial data

$$\phi(0,x) = f_\omega(x) + F(|x|), \qquad \phi_t(0,x) = i\omega f_\omega(x) + G(|x|).$$

Then if $\|(F,G)\|_{H^1 \times L^2}$ is sufficiently small the quantity

$$\inf_{\chi \in \mathbb{R}} \left(\|\phi(t,x) - e^{i\chi} e^{i\omega t} f_\omega(x)\|_{H^1} + \|\phi_t(t,x) - i\omega e^{i\chi} e^{i\omega t} f_\omega(x)\|_{L^2} \right) \tag{10}$$

remains uniformly bounded by a multiple of $\|(F,G)\|_{H^1 \times L^2}$ if ω lies in the stability interval. This provides a useful criterion for stability and in fact if ω does not satisfy this condition, instability was proved. Furthermore for the case of a pure power nonlinearity $\beta(|\phi|) = |\phi|^{p-1}$ the condition can be made completely explicit: a scaling argument gives

$$\mathbf{I} = \left\{ \omega \in \mathbb{R} : \frac{1}{1 + \frac{4}{p-1} - n} < \frac{\omega^2}{m^2} < 1 \right\},$$

which is nonempty when $1 < p < 1 + \frac{4}{n}$ and is empty otherwise ([4, 10]). This frequency dependence of the stability is one of the features which makes the problem difficult and in particular makes an understanding of stability at the linear level somewhat subtle as will be explained below.

The stability problem for general (nonradial) finite energy perturbations of (5) does not seem to have been investigated in the literature prior to [13], although the techniques of [4, 5] are certainly applicable in principle. The aim would be to obtain a uniform bound for

$$\inf_{\chi \in \mathbb{R}} \inf_{\xi \in \mathbb{R}^n} \left(\|\phi(t,x) - e^{i\chi} e^{i\omega t} f_\omega(x-\xi)\|_{H^1} + \|\phi_t(t,x) - e^{i\chi} i\omega e^{i\omega t} f_\omega(x-\xi)\|_{L^2} \right).$$

However, the technique of [4, 5] does not seem to give any information on the values of the parameters χ, ξ for which such an estimate might hold. The advantage of the method developed in [13] is that it yields explicit information on the time evolution of the parameters; this seems to be essential for studying the more general problem of constructing soliton type solutions for the equation (3) on a pseudo-Riemannian manifold. The new approach can be said to represent a "symplectification" of previous methods in the following sense. As well as the $n+1$ parameters χ, ξ the additional $n+1$ parameters ω, u are also allowed to vary (or "modulate"). This corresponds to fixing attention on the full $2n+2$ parameter family of solutions $\mathcal{S}$ which was described above. This turns out to be a *symplectic submanifold* in the *stable case* (see Theorem 3). Although as regards the stability analysis in the flat case the modulation of (ω, u) is not important, it does become important when there is a background metric which breaks the symmetry (and hence destroys the conservation laws which restrict the variation of (ω, u)).

3. Explicit approach to modulational stability

In this section the main theorem from [13] on modulational stability of (5) is stated in the pure power case, together with a brief description of some ideas used in the proof. As in the previous section the metric is the standard Minkowski one and we consider equation (4). The main result is Theorem 2, which also includes a precise definition of the notion of modulational stability. First of all it is necessary to write down explicitly the full set of solutions; the formulae represent nothing other than the action on (5) of the symmetry group G defined above.

Exact solutions. Consider the first order formulation of (4)

$$\begin{aligned} \partial_t \phi &= \psi, \\ \partial \psi &= \Delta \phi - m^2 \phi + \beta(|\phi|)\phi. \end{aligned} \tag{11}$$

This is a Hamiltonian evolution with Hamiltonian

$$H(\phi, \psi) = \frac{1}{2} \int_{\mathbb{R}^n} \Big(|\psi|^2 + |\nabla \phi|^2 + m^2 |\phi|^2 - \mathcal{V}(\phi) \Big) dx,$$

where $2\beta(|\phi|)\phi$ is the derivative of $\mathcal{V}$. The action of the Poincaré group and S^1 on the family (5), for $\omega \in (-m, m)$ gives a $2n+2$-parameter family $\mathcal{S}$ of solutions as follows. Let $\xi \in \mathbb{R}^n$ parametrise the action of $\mathbb{R}^n$ by translation, let $e^{i\theta}$ denote the action of S^1, and let $u \in \mathbb{R}^n$ with $|u| < 1$ parametrise the Lorentz transformation. Combine these parameters with ω into $\lambda = (\omega, \theta, \xi, u) \in O$, where $O \subset \mathbb{R}^{2n+2}$ is given by

$$O \equiv \{(\omega, \theta, \xi, u) \in \mathbb{R}^{2n+2} : |u| < 1 \text{ and } |\omega| < m\}. \tag{12}$$

Lorentz transformation in the direction $u \in \mathbb{R}^n$ involves a stretching by the Lorentz contraction factor

$$\gamma = \overline{\gamma}(u) \equiv (1 - |u|^2)^{-1/2} \tag{13}$$

in the direction of u only. So introduce the projection operator

$$\begin{aligned} P_u : \mathbb{R}^n &\to \mathbb{R}^n, \\ x &\mapsto u(x \cdot u)/|u|^2 \end{aligned}$$

in the direction of $u \in \mathbb{R}^n$ and let Q_u be its orthogonal complement:

$$Q_u + P_u = \mathbf{1}.$$

Now define

$$\mathbf{Z}(x; \lambda) = \gamma P_u (x - \xi) + Q_u (x - \xi), \tag{14}$$

$$\mathbf{\Theta}(x; \lambda) = \theta - \omega u \cdot \mathbf{Z}(x; \lambda). \tag{15}$$

Notice that $\mathbf{Z}$ (resp. $\mathbf{\Theta}$) are smooth functions from $\mathbb{R}^n \times O$ to $\mathbb{R}^n$ (resp. $\mathbb{R}$), and most importantly that *if $\xi = ut$ and $\theta = \omega t/\gamma$ then $\mathbf{Z}$ (resp. $\mathbf{\Theta}$) are the Lorentz transformations of x (resp. ωt).* Therefore the following lemma expresses the invariance of (4) under the group G:

Lemma 1. *Assume f_ω is a solution of (6). For $\lambda \in O$ define (ϕ_S, ψ_S) by*

$$\phi_S(x; \lambda) = e^{i\mathbf{\Theta}(x;\lambda)} f_\omega(\mathbf{Z}(x; \lambda)), \tag{16}$$

$$\psi_S(x; \lambda) = e^{i\mathbf{\Theta}(x;\lambda)} \big(i\omega\gamma f_\omega(\mathbf{Z}(x; \lambda)) - \gamma u \cdot \nabla_Z f_\omega(\mathbf{Z}(x; \lambda)) \big). \tag{17}$$

Then

$$\phi(t,x) = \phi_S(x;\lambda(t)), \;\; \psi(t,x) = \psi_S(x;\lambda(t))$$

is a solution to (11) *if* λ *evolves according to*

$$\frac{d\lambda}{dt} = V(\lambda), \tag{18}$$

where V *is the vector field on* O *given by*

$$V(\lambda) = V(\omega,\theta,\xi,u) = (0,\omega/\gamma,u,0). \tag{19}$$

The space of soliton solutions defined in the discussion following (7) is thus given by:

$$\mathcal{S} \equiv \big\{(\phi_S(\cdot,\lambda),\psi_S(\cdot,\lambda)) : \lambda \in O\big\} \subset H^1 \times L^2. \tag{20}$$

Statement of stability theorem. For simplicity the theorem will only be stated for the pure power case when $\beta(|\phi|) = |\phi|^{p-1}$. The aim is to prove stability with respect to finite energy perturbations of the Cauchy data of the solution

$$\big(\phi_S(\cdot;\lambda^{(0)}(t)),\psi_S(\cdot;\lambda^{(0)}(t))\big),$$

where $\lambda^{(0)}(t)$ verifies (18) with initial value $\lambda^{(0)}(0) = \lambda_0 = \big(\omega_0,\theta_0,\xi_0,u_0\big)$, as long as

$$\omega_0 \in \mathbf{I},$$

where the stability interval $\mathbf{I}$ is as defined in (9). The hypothesis $\omega_0 \in \mathbf{I}$ is thus nonvacuous if $1 < p < 1 + 4/n$.

Theorem 2 ([13]). *Consider the Cauchy problem for* (11), *with* $\beta(|\phi|) = |\phi|^{p-1}$, *and with initial values* $\big(\phi(0,\cdot),\psi(0,\cdot)\big) \in H^1 \oplus L^2$. *Define*

$$O_{stab} \equiv \{(\omega,\theta,\xi,u) \in O : \omega \in \mathbf{I}\}. \tag{21}$$

Then for all $\lambda_0 \in O_{stab}$ *there exists* $\epsilon_* = \epsilon_*(\lambda_0) > 0$ *such that if*

$$\epsilon = \|\phi(0,\cdot) - \phi_S(\cdot;\lambda_0)\|_{H^1} + \|\psi(0,\cdot) - \psi_S(\cdot;\lambda_0)\|_{L^2} < \epsilon_*(\lambda_0)$$

then there exist $\lambda \in C^1(\mathbb{R};O_{stab})$, *a pair* $(\phi,\psi) \in C(\mathbb{R};H^1 \oplus L^2)$ *and* $c_1 > 0$ *such that*

$$\sup_{t\in\mathbb{R}}\Big(\|\phi(t,\cdot) - \phi_S(\cdot;\lambda(t))\|_{H^1} + \|\psi(t,\cdot) - \psi_S(\cdot;\lambda(t))\|_{L^2}\Big) < c_1\epsilon. \tag{22}$$

The curve $t \mapsto \lambda(t)$ *is the solution of an explicitly determined system of ordinary differential equations and there exists* $c_2 > 0$ *such that*

$$|\partial_t\lambda - V(\lambda)| \le c_2\epsilon. \tag{23}$$

In this situation the exact solution $\big(\phi_S(x;\lambda^{(0)}(t)),\ \psi_S(x;\lambda^{(0)}(t))\big)$ *of* (11) *is said to be* modulationally stable.

Notice that since $\lambda_0 = \big(\omega_0,\theta_0,\xi_0,u_0\big)$ can be anywhere in O_{stab} this theorem asserts that stability holds for all velocities u_0 of the soliton. The fact that the stability criterion is independent of velocity is certainly to be expected in view of Lorentz invariance, although it is *not* possible to reduce study of the Cauchy problem for different velocities to that at zero because the Lorentz transform does not preserve the $t =$ const. hyperplanes. The assertion of stability for all velocities and without radial symmetry does not appear to have been stated explicitly in the literature before, although it should be derivable from the general theorems in [5]. As already remarked the main point of the new approach taken here is to give explicit information on the curve $t \mapsto \lambda(t)$. Indeed the system of equations satisfied by λ is precisely given in [13]. It is the corresponding set of equations in the case when curvature is present that leads to the geodesic equation in the proof of Theorem 7.

The analysis is similar to that in [19], where a linearised stability result for (5) in the nonlinear Schrödinger equation was obtained. Here there is the additional complication (not present in the Schrödinger case) caused by the dependence of the stability condition on ω (see (9)). Indeed this latter fact makes even the understanding of stability at the linear level somewhat subtle due to the following rescaling argument. In the pure power case $\beta(|\phi|)\phi = |\phi|^{p-1}\phi$ the equation (6) becomes

$$-\Delta f_\omega + (m^2 - \omega^2) f_\omega = f_\omega^p$$

so that it is easy to check that the solutions are all related by rescaling:

$$f_\omega(x) = (m^2 - \omega^2)^{\frac{1}{p-1}} f(\sqrt{m^2 - \omega^2}\, x), \tag{24}$$

where f is the corresponding solution with $m^2 - \omega^2 = 1$. Furthermore the linear operators which appear on linearisation, namely

$$\begin{aligned} L_+ &= -\Delta_Z + (m^2 - \omega^2) - p f_\omega^{p-1}(Z), \\ L_- &= -\Delta_Z + (m^2 - \omega^2) - f_\omega^{p-1}(Z) \end{aligned}$$

are also transformed into a universal form (independent of ω) under this rescaling. So at first sight it is unclear how the stability properties can change as ω changes. This apparent paradox is settled by Lemma 4 below.

Remarks on the proof. To start with write

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = \begin{pmatrix} \phi_S(x; \lambda(t)) + \tilde\phi(t) \\ \psi_S(x; \lambda(t)) + \tilde\psi(t) \end{pmatrix}.$$

The purpose is to determine $\lambda(t)$ so that the error terms $\tilde\phi$, $\tilde\psi$ can be estimated in $H^1 \oplus L^2$ uniformly in time. The crucial points for achieving this are:

- (i) to identify, at the linear level, the origin of the stability interval as a condition for positivity of the Hessian of an appropriate functional called the *augmented Hamiltonian* on an appropriate subspace which is *the symplectic normal subspace*;
- (ii) to show that there is a locally well-posed system of ordinary differential equations for $t \mapsto \lambda(t)$ which forces $(\tilde\phi, \tilde\psi) = (\phi - \phi_S, \psi - \psi_S)$ to lie in the subspace referred to in (i).

In order to fully explain these points, definitions of the italicised phrases in (i) are needed.

(a) *The augmented Hamiltonian.* It is useful to note, at the formal level, that (ϕ_S, ψ_S) are critical points of H subject to the constraints that Π, Q be fixed, where

$$\Pi_i(\phi, \psi) = \int \left\langle \psi, \frac{\partial \phi}{\partial x^i} \right\rangle dx, \qquad \text{(momentum)} \tag{25}$$

$$Q(\phi, \psi) = \int \langle i\psi, \phi \rangle dx \qquad \text{(charge)}. \tag{26}$$

These are the conserved quantities deriving from translation and phase (S^1) invariance on account of Noether's theorem. In this setting u^i and ω/γ emerge as the corresponding Lagrange multipliers and thus ϕ_S, ψ_S are critical points of the enlarged functional

$$F(\phi,\psi;\lambda) = H(\phi,\psi) + u^i\left(\Pi_i(\phi,\psi) - p_i\right) + \frac{\omega}{\gamma}\left(Q(\phi,\psi) - q\right), \tag{27}$$

for appropriate p, q: this quantity is called the augmented Hamiltonian. Thus the Hessian of F at $\left(\phi_S(\cdot,\lambda), \psi_S(\cdot,\lambda)\right)$ should be an important quantity for the stability analysis.

(b) *The symplectic normal subspace.*[4] Introduce the subspace

$$\Upsilon_\lambda \equiv \left\{(\tilde\phi,\tilde\psi) \in H^1 \oplus L^2 : \boldsymbol{\Omega}\Big((\tilde\phi,\tilde\psi), \frac{\partial}{\partial\lambda_A}(\phi_S,\psi_S)\Big) = 0 \text{ for all } A\right\}, \tag{28}$$

where $\boldsymbol{\Omega}$ is the standard symplectic form for wave equations,

$$\begin{aligned} &\boldsymbol{\Omega} : \Big(L^2(\Sigma;\mathbb{C})\Big)^2 \times \Big(L^2(\Sigma;\mathbb{C})\Big)^2 \to \mathbb{R} \\ &\boldsymbol{\Omega}\Big((\tilde\phi,\tilde\psi),(\phi',\psi')\Big) = \int \Big(\langle\tilde\phi,\psi'\rangle - \langle\tilde\psi,\phi'\rangle\Big)\,dx. \end{aligned} \tag{29}$$

Υ_λ is the symplectic normal subspace to $\mathcal{S} = \left(\phi_S(\cdot,\lambda),\psi_S(\cdot,\lambda)\right)\big|_{\lambda\in O}$.

Having made these definitions return now to a discussion of the two crucial points of the proof referred to above. Point (ii) is resolved by a lengthy but straightforward calculation relying on the following observation, which is the first intimation of the significance of the condition on ω in (9).

Lemma 3 ([13]). *The subset $\mathcal{S}_{stab} = \left(\phi_S(\cdot,\lambda),\psi_S(\cdot,\lambda)\right)\big|_{\lambda\in O_{stab}} \subset H^1 \times L^2$ is a local C^1 symplectic submanifold, in particular $\bar{\boldsymbol{\Omega}}$, the restriction of $\boldsymbol{\Omega}$ to $\mathcal{S}_{stab}$, is nondegenerate. As λ approaches the boundary of O_{stab} in O, i.e., as $\frac{\omega^2}{m^2}$ tends to $\frac{1}{1+\frac{4}{p-1}-n}$, $\bar{\boldsymbol{\Omega}}$ degenerates.*

The stability condition thus makes it possible to restrict the Hamiltonian flow (11) to $\mathcal{S}$ in a nondegenerate fashion: this is important for the well-posedness of the system of ODE's for λ referred to in (ii) above. Notice also that Lemma 3 justifies the terminology "symplectic normal subspace" in (b) above.

Point (i) is dealt with by the following *infinitesimal stability* lemma. For fixed $\lambda = (\omega,\theta,\xi,u) \in O$ consider the quadratic form on $H^1 \times L^2$:

$$\begin{aligned} \mathcal{E}(\tilde\phi,\tilde\psi;\lambda) = \frac{1}{2}\int \Big(&|\tilde\psi|^2 + |(\nabla_x)_i\tilde\phi|^2 + m^2|\tilde\phi|^2 - \beta(|\phi_S|)|\tilde\phi|^2 \\ &- \beta'(|\phi_S|)\frac{\langle\phi_S,\tilde\phi\rangle^2}{|\phi_S|} - \frac{2\omega}{\gamma}\langle\tilde\psi, i\tilde\phi\rangle + 2u^i\langle(\nabla_x)_i\tilde\phi,\tilde\psi\rangle\Big)\,dx, \end{aligned}$$

where $\phi_S = \phi_S(x;\lambda)$ is as above in (16) ($\mathcal{E}$ is independent of ψ_S because the nonlinearity appears only in ϕ). The significance of the quantity $\mathcal{E}$ is that it is the Hessian of the augmented Hamiltonian F.

Lemma 4 ([13]). *The quadratic form $\mathcal{E}(\tilde\phi,\tilde\psi;\lambda)$, restricted to the subspace Υ_λ, is equivalent to the $H^1 \oplus L^2$ norm, uniformly for λ in compact subsets of O_{stab}.*

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[4] Υ_λ is the space of vectors Ω-orthogonal to the tangent space to $\mathcal{S}$. The terminology is justified in the stable case because $\mathcal{S}$ is then symplectic.

This information can be combined with the conservation laws for H, Π, Q to imply a uniform $H^1 \times L^2$ bound for $(\tilde{\phi}, \tilde{\psi})$ in the stable case and thus prove Theorem 2.

More general nonlinearities. The class of nonlinearities $\mathcal{F}(\phi) = \beta(|\phi|)\phi$ to which the present development can be extended is delineated primarily by the requirement that Lemma 4 holds. This is essentially ensured by the conditions for uniqueness of positive f_ω (see [8] and references therein) which guarantee that the subspace on which $\mathcal{E}$ is negative is one-dimensional. (It is of course also necessary that f_ω exist, and that the Cauchy problem is well posed, so that it is always assumed that the nonlinearity is subcritical.)

In the pure power case Lemma 4 can be deduced from the fact ([18]) that f_ω are optimizers of the Gagliardo-Nirenberg quotient:

$$J^{p,n}(u) \equiv \frac{\|\nabla u\|_{L^2}^{\frac{(p-1)n}{2}} \|u\|_{L^2}^{2+\frac{(p-1)(2-n)}{2}}}{\|u\|_{L^{p+1}}^{p+1}}. \tag{30}$$

This is related to the scaling and invariance properties of (6) in the pure power case (also see (24) and surrounding discussion) and allows an explicit determination of the stability interval (9). For more general nonlinearities Lemma 4 can be proved under the assumptions just mentioned using (9) as a characterisation of the stability interval $\mathbf{I}$. However, unlike the pure power case it may not be possible to delineate $\mathbf{I}$ any more explicitly than this.

4. Nontopological solitons on pseudo-Riemannian manifolds

In this section I explain a special case of a result from [14] which generalises the techniques described in §3 to the wave equation on a manifold. So fix $n = 3$ and introduce on $\mathbb{M} = \mathbb{R}^{1+3}$ a pseudo-Riemannian structure $\underline{g}$ so that $(\mathbb{M}, \underline{g})$ is a pseudo-Riemannian manifold, and assume there is a global system of coordinates $(x^0 = t, x^1, x^2, x^3)$ in which the metric is of the form[5]

$$ds^2 = \underline{g}_{\mu\nu} dx^\mu dx^\nu = \left(-p^2 dt^2 + g_{ij} dx^i dx^j\right). \tag{31}$$

$(\mathbb{M}, \underline{g})$ admits a t-foliation into space-like hypersurfaces which are the level sets of the time function $t : \mathbb{M} \to \mathbb{R}$, i.e.,

$$\mathbb{M} \approx \mathbb{R} \times \Sigma, \qquad \Sigma \approx \mathbb{R}^n, \qquad \Sigma_{t_0} \equiv \{t_0\} \times \Sigma = t^{-1}(t_0). \tag{32}$$

In (31) $p : \mathbb{M} \to \mathbb{R}$ is called the *lapse* function and $g_{ij}(t, x)$ is the metric induced on Σ_t from ds^2. It is assumed that p, g are C^2 and there exists a number $K > 1$ such that[6]

- all derivatives up to second order of p^2 and g_{ij} are bounded by K;
- p^6 and $\mathbf{g} = \det g_{ij}$ are bounded below by $\frac{1}{K^3}$.

As usual the induced inner product on the cotangent space is represented by the inverse matrix $g^{ij}(t, x)$ and the notation $\mathbf{g} = \det g_{ij}$ will be used throughout.

[5] The summation convention will be used. Greek (space-time) indices run from 0 to 3, and Latin (space) indices run from 1 to 3.

[6] The exponents are chosen to make the assumptions scale invariant.

A scaling limit. The aim is to construct solutions to the wave equation on $(\mathbb{M}, \underline{g})$ using the flat Minkowski space solutions in Lemma 1 as basic building blocks. This is possible in a certain scaling limit in which the size of the soliton is small compared to length scales over which the metric varies. It is worthwhile to introduce this scaling first in the flat case: here it corresponds to rescaling the metric

$$-dt^2 + \sum_{i=1}^{3}(dx^i)^2 \Rightarrow \frac{1}{\epsilon^2}\Big(-dt^2 + \sum_{i=1}^{3}(dx^i)^2\Big).$$

Under this scaling the equation (4) becomes

$$\partial_t^2\phi - \Delta\phi + m^2\phi = \frac{1}{\epsilon^2}\beta(|\phi|)\phi. \tag{33}$$

The basic soliton solution is thus now

$$\phi(t,x) = e^{i\omega t/\epsilon} f_\omega(x/\epsilon). \tag{34}$$

Recalling that f has exponential decay at rate m it follows that the soliton is now exponentially localised in a region of size ϵ. Now in the presence of a nontrivial metric consider the rescaling:

$$\underline{g} \Rightarrow \frac{1}{\epsilon^2}\underline{g} = \frac{1}{\epsilon^2}\Big(-p^2 dt^2 + g_{ij}dx^i dx^j\Big),$$

where p, g are fixed (independent of ϵ) as above. In the limit $\epsilon \longrightarrow 0$ the size of the soliton will be small compared to distances over which p, g vary.

Under this rescaling the corresponding Lebesgue and Sobolev spaces on $\Sigma \approx \mathbb{R}^n$ defined with the metric $\frac{1}{\epsilon^2}g(t,\cdot)$ are denoted by L^2_ϵ and H^1_ϵ, and their norms analogously:

$$\begin{aligned} \|f\|^2_{L^2_\epsilon} &= \epsilon^{-n}\int_\Sigma |f(x)|^2\sqrt{\mathrm{g}}\,dx, \\ \|f\|^2_{H^1_\epsilon} &= \int_\Sigma \big(\epsilon^{2-n} g^{jk}\nabla_j f(x)\nabla_k f(x) + \epsilon^{-n}|f(x)|^2\big)\sqrt{\mathrm{g}}\,dx. \end{aligned} \tag{35}$$

The assumptions made on the metric g ensure that the norms so defined, which are different for different t, are in fact uniformly equivalent with constant depending upon K only, so there is no need to distinguish between them for the purposes of this paper.

The wave equation on $(\mathbb{M}, \underline{g})$**.** With the metric $\epsilon^{-2}\underline{g}$ the wave equation (4) generalises to

$$\partial_t\left(\frac{\sqrt{\mathrm{g}}}{p}\partial_t\phi^\epsilon\right) - \partial_i(p\sqrt{\mathrm{g}}g^{ij}\partial_j\phi^\epsilon) + \frac{1}{\epsilon^2}p\sqrt{\mathrm{g}}(m^2\phi^\epsilon - \beta(|\phi^\epsilon|))\phi^\epsilon = 0. \tag{36}$$

From this it is again clear that ϵ controls the size of the soliton compared to length scales over which the metric varies: the size of the soliton is $O(\epsilon)$ as $\epsilon \longrightarrow 0$, while p, g vary over scales of $O(1)$.

Geodesics on $(\mathbb{M}, \underline{g})$**.** To describe time-like geodesics on $(\mathbb{M}, \underline{g})$ it is convenient to parametrise them with the time coordinate t. So using the foliation structure in (32) this gives a curve $t \mapsto (t, \xi(t)) \in M = \mathbb{R}\times\Sigma$ and hence a curve $t \mapsto \xi(t) \in \Sigma$. To write down the conditon for this curve to be a geodesic introduce the velocity

$$u^j = \frac{d\xi^j}{dt}.$$

At fixed t this is a vector in $T_{\xi(t)}\Sigma$. Introduce also the evaluations of the metric coefficients along the curve:

$$
\begin{aligned}
q(t) &= p(t,\xi(t)), & q_{,k}(t) &= \frac{\partial p}{\partial x^k}(t,\xi(t)),\\
h_{ij}(t) &= g_{ij}(t,\xi(t)), & h^{ij}_{\ ,k}(t) &= \frac{\partial g^{ij}}{\partial x^k}(t,\xi(t)).
\end{aligned} \tag{37}
$$

Since $u(t) \in T_{\xi(t)}\Sigma$, operations to raise/lower indices for u and compute the norm are carried out with $h_{ij}(t)$:

$$u_j(t) = h_{jk}(t)u^k(t), \qquad |u|_h^2 = h_{jk}u^j u^k.$$

In terms of this the Lorentz contraction factor is

$$\gamma = \gamma(t) = \overline{\gamma}(q^{-1}|u(t)|_h)$$

(where $\overline{\gamma}(s) \equiv (1-s^2)^{-1/2}$ as above). In terms of u and γ the geodesic equation takes the form

$$\frac{d}{dt}\left(\frac{\gamma u_k}{q}\right) + \gamma q_{,k} + \frac{\gamma u_i u_j}{2q} h^{ij}_{\ ,k} = 0. \tag{38}$$

In terms of ξ this is a second order nonlinear differential equation.

Statement of theorem on geodesic hypothesis. The objective is to construct solutions of (36) which "look like" (5) centred at a point $\xi(t)$ at each time t. In view of the discussion of the geodesic hypothesis in §1 it is to be anticipated that this will only be possible if $t \mapsto \xi(t)$ is close to being a geodesic, i.e. a solution of (38). To achieve and make precise this objective I will now introduce, given $\epsilon > 0$, a function $\phi_0^\epsilon : \mathbb{M} \to \mathbb{C}$ which (by definition) represents a nontopological soliton centred on the curve $\xi(t)$. The idea is to "freeze" the metric coefficient functions at their values at the centre $\xi(t)$ as in (37) and regard these as defining a constant coefficient metric. But in the constant coefficient case the formulae (14)–(17) give exact solutions (after making a linear change of variables to put the metric in standard Minkowski form). It is these formulae which motivate the function introduced in the following definition. The fact that this process of "freeezing the coefficients" has any validity of course has to be proved. It is only expected to be a good idea in the scaling limit in which the soliton is small so that it only "sees" the values of the coefficients in a small neighbourhood of the point $\xi(t)$. It is the purpose of Theorem 7 to justify this expectation.

Corresponding to the projection operator in the direction of u which appears in the Lorentz transformation formulae (14)–(15) it is now necessary to introduce the projection operator $P_t : \mathbb{R}^n \to \mathbb{R}^n$ along $u(t)$ *defined with respect to the inner product* $h(t)$. The operator P_t and its orthogonal complement Q_t are given explicitly by:

$$(P_t V)^i = \frac{\langle u, V\rangle_h u^i}{|u|_h^2}, \tag{39}$$

$$
\begin{aligned}
(Q_t V)^i &= V^i - \left(P_t V\right)^i\\
&= \frac{|u|_h^2 V^i - \langle u, V\rangle_h u^i}{|u|_h^2},
\end{aligned} \tag{40}
$$

where the t dependence of h_{ij} and u is suppressed.

Definition 5. Given $(\omega,\eta,\xi,u)\in C([0,t_1];\mathbb{R}^2\times T\Sigma)$ and $\epsilon>0$, define a nontopological soliton on $\big(\mathbb{M},\epsilon^{-2}\underline{g}\big)$ centred on the curve $\xi(t)$, with frequency $\omega(t)$ and phase $\eta(t)$, to be the function $\phi_0^\epsilon:\mathbb{M}\to\mathbb{C}$ given by

$$\phi_0^\epsilon(t,x)=f_\omega\Big(\frac{1}{\epsilon}|\gamma P_t(x-\xi)+Q_t(x-\xi)|_h\Big)\exp\Big[\frac{i}{\epsilon}\Big(\int_0^t\frac{\omega q}{\gamma}dt'+\eta-\frac{\gamma}{q}\langle x-\xi,u\rangle_h\Big)\Big].$$

(In this formula the t dependence of $\omega,\eta,\xi,u,h,q,\gamma$ is supressed. The notation $|\cdot|_h$ and $\langle\cdot\rangle_h$ is used for, respectively, the norm and inner product defined by h_{ij} as above.)

Remark 6. As remarked above, the origin of this formula is as a change of variables from (14)–(17) obtained by freezing the metric coefficients. Another helpful way to understand the formula is to use the coordinate system $(\hat t,\hat x)$ adapted to an observer moving along the curve $t\mapsto\xi(t)$ (see [9, §13.6]): in this system of coordinates the function $\phi_0^\epsilon\sim e^{i\omega\hat t}f_\omega(|\hat x|)$ as explained in [14].

Again it is important to emphasize that the function in Definition 5 is not an exact solution of (36) in the nonflat case. Nevertheless the following theorem indicates that there does exist a solution close to it when $\xi(t)$ is close to a geodesic.

Theorem 7 ([14]). *Let $n=3$ and $\beta(|\phi|)=|\phi|^{p-1}$ with $2\le p<1+4/n=7/3$, so that in particular the stability interval $\mathbf{I}$ defined in (9) is nonempty. Given $\omega(0)\in\mathbf{I}$ and a time-like geodesic*

$$t\mapsto\xi^{(0)}(t)\in\Sigma,$$

there exist

(i) *positive numbers $c_*>0$, $t_*>0$, $\epsilon_*>0$,*
(ii) *a C^1 function $(\omega,\eta,\xi,u)\in C^1([0,t_*];\mathbb{R}^2\times T\Sigma)$, and*
(iii) *initial data $(\phi^\epsilon(0),\phi_t^\epsilon(0))$, for $0<\epsilon<\epsilon_*$,*

such that for all $0\le t\le t_*$ and $0<\epsilon<\epsilon_*$,

$$\epsilon^{-2}|\omega(t)-\omega(0)|+\epsilon^{-2}|\eta(t)-\eta(0)|+\epsilon^{-1}|\xi(t)-\xi^{(0)}(t)|+\epsilon^{-1}\Big|u(t)-\frac{d\xi^{(0)}}{dt}\Big|\le c_*,$$

while the solution ϕ^ϵ to (36) satisfies, for all $0\le t\le t_*$ and $0<\epsilon<\epsilon_*$, the estimate

$$\big\|\phi^\epsilon(t,x)-\phi_0^\epsilon(t,x)\big\|_{H^1_\epsilon}+\epsilon\big\|\partial_t\big(\phi^\epsilon(t,x)-\phi_0^\epsilon(t,x)\big)\big\|_{L^2_\epsilon}\le c_*\epsilon,$$

where $H^1_\epsilon, L^2_\epsilon$ are the scaled norms in (35) and ϕ_0^ϵ is as above, determined by the curve $(\omega,\eta,\xi,u)\in C^1([0,t_*];\mathbb{R}^2\times T\Sigma)$, whose existence is asserted in (ii).

Remark 8. The condition $\omega(0)\in\mathbf{I}$ is crucial because it determines the stability of (5) as discussed above. The upper limit $p<1+4/n$ ensures that the interval $\mathbf{I}$ is nonempty, while the lower limit $p\ge 2$ is for regularity purposes, and could possibly be relaxed. More significantly Theorem 7 is only a very special case of a general theorem proved in [14] for a large class of nonlinearities in arbitrary space dimension.

Remark 9. In terms of the standard (unscaled) L^2 norm the estimate in Theorem 7 can be written:

$$\epsilon^{-\frac{n}{2}}\Big(\big\|\nabla_x\big(\phi^\epsilon(t,x)-\phi_0^\epsilon(t,x)\big)\big\|_{L^2}+\big\|\partial_t\big(\phi^\epsilon(t,x)-\phi_0^\epsilon(t,x)\big)\big\|_{L^2}\Big)\\ +\epsilon^{-1-\frac{n}{2}}\big\|\big(\phi^\epsilon(t,x)-\phi_0^\epsilon(t,x)\big)\big\|_{L^2}\le c_*.$$

5. Further generalisations

In this final section I briefly mention some generalisations of Theorems 2 and 7.

Modulational stability in general Hamiltonian systems. The problem discussed in [13] is one of a large class of stability problems arising for Hamiltonian systems with symmetry for which a very general theory was developed in [4, 5]. It would be worthwhile to extend the method in [13] to prove general *modulational* stability results (in the sense of Theorem 2) for Hamiltonian systems, in particular Hamiltonian PDE's. The abstract setting is this: there is a submanifold $\mathcal{S}$ of the phase space foliated by integral curves of a vector field which are solutions of the original Hamiltonian equations. In the case that $\mathcal{S}$ is a *symplectic* submanifold with a tubular neighbourhood given by the symplectic normal bundle it is possible to make a decomposition of the solution (at each time) into a soliton, i.e. a point in $\mathcal{S}$, and a deformation which lies in the symplectic normal subspace. The stability condition of [4, 5] is then used to ensure the positivity of the Hessian of the augmented Hamiltonian as a quadratic form *on the symplectic normal subspace.* This can then be combined with the conservation laws to derive stability theorems which generalise Theorem 2 to an abstract setting. General results in this direction will be presented in [15].

Einstein's equation and the geodesic hypothesis. In the work [14] summarised in §4 I have considered the problem of motion of nontopological solitons in the semilinear wave equation on a manifold $(\mathbb{M}, \underline{g})$. Thus $\underline{g}$ has so far been a fixed background metric, corresponding to the notion of an external, or applied, gravitational field. In the general theory of relativity the metric $\underline{g}$ becomes a dynamical variable itself and evolves according to Einstein's equation:

$$R_{\mu\nu}(\underline{g}) - \frac{1}{2}R(\underline{g})\underline{g}_{\mu\nu} = T_{\mu\nu}(\phi, \underline{g}). \tag{41}$$

Here $R_{\mu\nu}(\underline{g})$ is the Ricci curvature of the metric $\underline{g}$ and its trace $R(\underline{g})$ is the scalar curvature. The right hand side $T_{\mu\nu}$ is the stress-energy tensor; its precise form need not be given here. Einsten's equation is to be solved in conjunction with (3): together they form a quasilinear system which is essentially hyperbolic (modulo the usual proviso regarding gauge invariance). In this situation it is to be expected that the geodesic hypothesis is still valid as long as the amplitude of the soliton is small enough that its effect on the metric $\underline{g}$ can be treated perturbatively. (This assumption is in addition to the assumption already introduced that the size of the soliton is small compared to other length scales in the problem.) It is possible to arrange for the soliton to have small amplitude by taking advantage of the freedom to choose the nonlinear term $\mathcal{F}$ in (3). It is then possible to prove an analogue of Theorem 7 for the Einstein semilinear wave system comprising (3) and (41): such a theorem will be presented in [16].

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