ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 7, Pages 8–16 (April 2, 2001)
S 1079-6762(01)00089-0

RELATIVE ZETA DETERMINANTS AND THE GEOMETRY OF THE DETERMINANT LINE BUNDLE

SIMON SCOTT

(Communicated by Michael Taylor)

ABSTRACT. The spectral ζ-function regularized geometry of the determinant line bundle for a family of first-order elliptic operators over a closed manifold encodes a subtle relation between the local family's index theorem and fundamental non-local spectral invariants. A great deal of interest has been directed towards a generalization of this theory to families of elliptic boundary value problems. We give here precise formulas for the relative zeta metric and curvature in terms of Fredholm determinants and traces of operators over the boundary. This has consequences for anomalies over manifolds with boundary.

0. INTRODUCTION

In this paper we study the zeta-function regularized geometry of the determinant line bundle for a family of first-order elliptic boundary value problems (EBVPs). By an EBVP we mean an elliptic differential operator over a compact manifold with boundary, endowed with a global (injectively elliptic) boundary condition. In [8] it was shown that the determinant line bundle for such a family has a natural Hermitian metric defined by the Fredholm determinant of a canonically associated operator over the boundary. The resulting canonical metric has an analytical status which is essentially opposite to that of the usual zeta function metric [1, 7]. That the two metrics, and their associated connections, are nevertheless precisely related derives from a certain 'relativity principle for determinants', which asserts that for preferred classes of unbounded operators, ratios of ζ-determinants can be written canonically in terms of Fredholm determinants (Theorem 1). This is interesting because the (usual) ζ-determinant, on the other hand, does *not* define an extension of the Fredholm determinant (operators with Fredholm determinants do not have ζ-determinants)[1]; indeed there is no extension of the Fredholm determinant to a (multiplicative) determinant on general classes of elliptic pseudodifferential operators (ψdos).

This fact leads to the construction of 'higher' differential geometries for families of ψdos, via regularized traces and determinants, such as that defined by the spectral ζ-function. Our results indicate that the 'relativity principle' governs a precise relation between relative higher differential geometries and the (usual) canonical differential geometry associated to the standard trace. Applied to families of EBVPs

Received by the editors December 15, 1999, and, in revised form, September 15, 2000.
2000 *Mathematics Subject Classification.* Primary 58G20, 58G26, 11S45; Secondary 81T50.
[1] We consider here infinite-dimensional Hilbert spaces.

this means that geometric anomalies, associated with the break down of symmetry invariance in higher traces, are located on the boundary. Details of the constructions here, which build on ideas in [2, 3, 4, 5, 8, 10], will be presented in [9].

1. Relative zeta determinants

Let A_1, A_2 be invertible closed operators on a Hilbert space H with a common spectral cut $R_\theta = \{re^{i\theta} : r \geq 0\}$. This means that there exists an $\varepsilon > 0$ such that the resolvents $(A_i - \lambda)^{-1}$ are holomorphic in the sector

$$\Lambda_{\theta,\varepsilon} = \{z \in \mathbb{C}\backslash\{0\} \mid |\arg(z) - \theta| < \varepsilon\}$$

and such that the operator norms $\|(A_i - \lambda)^{-1}\|$ are $O(|\lambda|^{-1})$ as $\lambda \to \infty$ in $\Lambda_{\theta,\varepsilon}$. For $\Re(s) > 0$ one has the complex power operators

$$A_i^{-s} = \frac{i}{2\pi} \int_\Gamma \lambda_\theta^{-s} (A_i - \lambda)^{-1} \, d\lambda,$$

where $\lambda_\theta^{-s} = |\lambda|^{-s} e^{-is\arg(\lambda)}$, $\theta - 2\pi \leq \arg(\lambda) \leq \theta$, is the branch of λ^{-s} defined by the spectral cut R_θ, and Γ is the contour traversing R_θ from ∞ to a small circle around the origin, clockwise around the circle, and then back to ∞ along R_θ.

If we assume that the A_i^{-s} are trace class in some half-plane $\Re(s) > r > 0$, one then has spectral zeta functions

$$\zeta_\theta(A_i, s) = \operatorname{Tr} A_i^{-s} = \sum_{\mu \in \operatorname{sp}(A_i)} \mu^{-s}, \qquad \Re(s) > r.$$

We further assume that $\partial_\lambda^m (A_i - \lambda)^{-1}$ are trace class for $m+1 > r$ with asymptotic expansions as $\lambda \to \infty$ along the ray R_θ

$$\operatorname{Tr}\left(\partial_\lambda^m (A_i - \lambda)^{-1}\right) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{n_j} a_{j,k}^{(i)} (-\lambda)^{-(\alpha_j + m)} \log^k(-\lambda) \tag{1.1}$$

with $0 < \alpha_j + m \nearrow +\infty$. From Seeley's work (see also [4]) it is well known that the expansion (1.1) defines a meromorphic continuation of $\zeta_\theta(A_i, s)$ to all of $\mathbb{C}$, the term with coefficient $a_{j,k}^{(i)}$ corresponding to a pole of $\Gamma(s)\zeta_\theta(A_i, s)$ at $s = \alpha_j - 1$ of order $k+1$. In particular, if $a_{J,k}^{(1)} = a_{J,k}^{(2)} = 0$, where $\alpha_J = 1$ with $k \geq 1$, then there is no pole at $s = 0$ and one has the ζ-determinants

$$\det_{\zeta,\theta} A_1 = e^{-\zeta_\theta'(A_1, 0)}, \qquad \det_{\zeta,\theta} A_2 = e^{-\zeta_\theta'(A_2, 0)},$$

where $\zeta_\theta' = d/ds(\zeta_\theta)$. In this case we refer to each of A_1, A_2 as *ζ-admissible*.

We refer to (A_1, A_2) as *ζ-comparable* if

$$a_{j,k}^{(1)} = a_{j,k}^{(2)} \quad \text{for } j < J, \tag{1.2}$$

and

$$a_{J,k}^{(1)} = a_{J,k}^{(2)} \quad \text{for } k \geq 1,$$

and if the relative resolvent $(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}$ is trace class such that

$$\operatorname{Tr}\left((A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right) = -\frac{\partial}{\partial\lambda} \log \det{}_F \mathcal{S}_\lambda.$$

Here the 'scattering' operator $\mathcal{S}_\lambda = \mathcal{S}_\lambda(A_1, A_2)$ is an operator of the form $Id + W_\lambda$ on a Hilbert space $H' \subseteq H$ with W_λ of trace class, so that $\mathcal{S}_\lambda$ has a Fredholm determinant $\det_F \mathcal{S}_\lambda := 1 + \sum_{k\geq 1} \mathrm{Tr}\,(\bigwedge^k W_\lambda)$. The *relative spectral ζ-function*

$$\zeta_\theta(A_1, A_2, s) = \mathrm{Tr}\,(A_1^{-s} - A_2^{-s}) = \frac{i}{2\pi}\int_\Gamma \lambda_\theta^{-s}\,\mathrm{Tr}\,((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1})\,d\lambda$$

is then well defined and holomorphic in s for $\Re(s) > 0$, and equal to $\zeta_\theta(A_1, s) - \zeta_\theta(A_2, s)$. Using (1.2), we see that as $\lambda \to \infty$ in $\Lambda_{\theta,\varepsilon}$, there is an asymptotic expansion

$$\mathrm{Tr}\,((A_1-\lambda)^{-1} - (A_2-\lambda)^{-1}) \sim \sum_{j=J+1}^{\infty}\sum_{k=0}^{n_j} c_{j,k}(-\lambda)^{-\alpha_j}\log^k(-\lambda) + \frac{c_{J,0}}{\lambda}\,.$$

The meromorphic extension of $\zeta_\theta(A_1, A_2, s)$ to $\mathbb{C}$ is therefore regular at zero and we can define the *relative ζ-determinant* $\det_{\zeta,\theta}(A_1, A_2) = e^{-\zeta_\theta'(A_1,A_2,0)}$.

The *regularized limit* $\mathrm{LIM}^\theta_{\lambda\to\infty}$ of a function with an asymptotic expansion

$$f(\lambda) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{m_j} b_{j,k}(-\lambda)^{-\beta_j}\log^k(-\lambda)$$

as $\lambda \to \infty$ in $\Lambda_{\theta,\varepsilon}$, where $\beta_j \nearrow +\infty$ and $\beta_N = 0$, is defined to be the constant term in the expansion $b_{N,0}$. We have (with $\mathcal{S} := \mathcal{S}_0$):

Theorem 1. *For ζ-comparable operators A_1, A_2*

$$\det_{\zeta,\theta}(A_1, A_2) = \det_F\mathcal{S}\,.\,e^{-\mathrm{LIM}^\theta_{\lambda\to\infty}\log\det_F \mathcal{S}_\lambda}\,. \tag{1.3}$$

If A_1, A_2 are ζ-admissible, $\det_{\zeta,\theta}(A_1, A_2) = \det_{\zeta,\theta}A_1/\det_{\zeta,\theta}A_2\,.$

Proof. Since $\lambda^{-s}\log\det_F \mathcal{S}_\lambda \to 0$ at the ends of Γ for $\Re(s) > 0$, we can integrate by parts in

$$\zeta_\theta(A_1, A_2, s) = -\frac{i}{2\pi}\int_\Gamma \lambda_\theta^{-s}\,\partial_\lambda\log\det_F\mathcal{S}_\lambda\,d\lambda\,.$$

Taking the s derivative we end up with $\zeta_\theta'(A_1, A_2, s) = d/ds_{|s=0}(sg(s))$, where

$$g(s) = -\frac{i}{2\pi}\int_\Gamma \lambda_\theta^{-s}\,(\log\det_F\mathcal{S}_\lambda)/\lambda\,d\lambda$$

has a simple pole at $s = 0$. From the Laurent expansion of $(\log\det_F\mathcal{S}_\lambda)/\lambda$ around 0, and its asymptotic expansion as $\lambda \to \infty$ in $\Lambda_{\theta,\varepsilon}$, we can use the methods of [4] to obtain the full pole structure of $\zeta_\theta'(A_1, A_2, s)$ on $\mathbb{C}$. Evaluation at $s = 0$ then yields (1.3). The final statement follows from $\zeta_\theta(A_1, A_2, s) = \zeta_\theta(A_1, s) - \zeta_\theta(A_2, s)$. □

In the case $\zeta_\theta(A_1, A_2, 0) = 0$ the regularized LIM can be replaced by the usual lim, and $\zeta_\theta(A_1, A_2, s)$ exists at 0 without continuation. In particular, this applies to determinant class operators, that is, for A_1, A_2 with Fredholm determinants. The $\zeta_\theta(A_i, s)$ are then undefined for all s, but A_1, A_2 are always ζ-comparable and from (1.3)

$$\det_{\zeta,\theta}(A_1, A_2) = \det_F(A_2^{-1}A_1) = \frac{\det_F A_1}{\det_F A_2}. \tag{1.4}$$

Thus for any A of determinant class, $\det_{\zeta,\theta}(A, Id) = \det_F(A)$. This is independent of θ, equivalent to the fact that Γ can be closed at ∞ and replaced by a bounded contour.

1.1. **Elliptic boundary value problems.** Let X be a compact connected Riemannian manifold with boundary $\partial X = Y$ and let E^1, E^2 be Hermitian vector bundles over X. Let $A : C^\infty(X, E^1) \to C^\infty(X, E^2)$ be a first-order elliptic operator of Dirac type. By this we mean that there is a collar neighborhood $U = [0, 1) \times Y$ of the boundary in which A has the form

$$A_{|U} = \sigma\left(\frac{\partial}{\partial u} + B + R\right), \tag{1.5}$$

where $B : C^\infty(Y, E^1_{|Y}) \to C^\infty(Y, E^1_{|Y})$ is a first-order selfadjoint elliptic operator over the closed manifold Y, R is an operator of order 0, and $\sigma : E^1_{|U} \to E^2_{|U}$ a unitary isomorphism constant in u.

For each real $s > 1/2$, restriction to the boundary defines a continuous operator $\gamma : H^s(X, E^1) \to H^{s-1/2}(Y, E^1_{|Y})$ on the Sobolev completions, and we have the Cauchy data space $H(A, s) = \gamma \mathrm{Ker}(A, s)$, where

$$\mathrm{Ker}(A, s) = \{\psi \in H^s(X, E^1) \mid A\psi = 0\}.$$

Because of the Unique Continuation Property, $\gamma : \mathrm{Ker}(A, s) \to H(A, s)$ is a bijection while the *Poisson operator* $\mathcal{K}_A : H^{s-1/2}(Y, E^1_{|Y}) \longrightarrow \mathrm{Ker}(A, s) \subset H^s(X, E^1)$ of A defines a left inverse to γ.

The classical pseudodifferential operator (ψdo) $P(A) := \gamma\mathcal{K}_A$ of order 0 is a projection on $H^{s-1/2}(Y, E^1_{|Y})$ with range $H(A, s)$, called the *Calderón projection*. Associated to $P(A)$ we have the *pseudodifferential Grassmannian* $Gr_{-1}(A)$ parameterizing (orthogonal) projections P on $H_Y = L^2(Y, E^1_{|Y})$ such that $P - P(A)$ is a ψdo of order -1. The *smooth Grassmannian* $Gr_{-\infty}(A)$ is the infinite-dimensional dense submanifold of $Gr_{-1}(A)$ of those P such that $P - P(A)$ is a smoothing operator. Each $P \in Gr_{-1}(A)$ defines an EBVP

$$A_P = A : \mathrm{dom}\,(A_P) \longrightarrow L^2(X, E^2),$$

with $\mathrm{dom}\,(A_P) = \{\psi \in H^1(X, E^1) \mid P\gamma\psi = 0\}$. As a Fredholm operator, A_P is modeled by the boundary operator $S_A(P) = P \circ P(A) : H(A) \to W = \mathrm{range}\,(P)$, in so far as the Poisson operator effects canonical isomorphisms $\mathrm{Ker}A_P \cong \mathrm{Ker}S_A(P)$, $\mathrm{Coker}A_P \cong \mathrm{Coker}S_A(P)$, leading to the relative-index formulas

$$\mathrm{ind}\,(A_{P_1}) - \mathrm{ind}\,(A_{P_2}) = \mathrm{ind}\,(P_2, P_1), \qquad \mathrm{ind}\,(A_P) = \mathrm{ind}\,(S_A(P)),$$

where $(P_i, P_j) := P_j \circ P_i : \mathrm{range}\,(P_i) \to \mathrm{range}\,(P_j)$. Moreover, if A_P is invertible, then so is $S_A(P)$ and we can then define the Poisson operator of A_P by

$$\mathcal{K}_A(P) := \mathcal{K}_A S_A(P)^{-1} P : H^{s-1/2}(Y, E^1_{|Y}) \longrightarrow H^s(X, E^1).$$

From the identities $A_P^{-1}A = I - \mathcal{K}_A(P)\gamma$ and $A_P A_P^{-1} = Id_{L^2}$ we obtain the relative-inverse formula

$$A_{P_1}^{-1} = A_{P_1}^{-1} A_{P_2} A_{P_2}^{-1} = (A_{P_1}^{-1}A)A_{P_2}^{-1} = A_{P_2}^{-1} - \mathcal{K}_A(P_1)\gamma A_{P_2}^{-1}. \tag{1.6}$$

For references to details of these facts we refer the reader to [3, 4, 10].

Proposition 1.1. *Let A_z be a 1-parameter family of Dirac-type operators depending smoothly on a complex parameter z. Let $\dot{A}_z = \frac{d}{dz}A_z$ and let $P_1, P_2 \in Gr_{-1}(A)$ such that $P_1 - P_2$ has a smooth kernel and such that A_{z,P_i} are invertible for each z. Then $\dot{A}_z(A_{z,P_1}^{-1} - A_{z,P_2}^{-1})$ is a trace class operator on $L^2(X, E_2)$ with*

$$\mathrm{Tr}\,(\dot{A}_z(A_{z,P_1}^{-1} - A_{z,P_2}^{-1})) = \frac{d}{dz}\log\det{}_F\left(\frac{S_z(P_1)}{P_1S_z(P_2)}\right). \tag{1.7}$$

[For simplicity we assume here that (P_2, P_1) is invertible. For the general case see [9].*]*

Proof. We compute that

$$\frac{d}{dz}A_z.\mathcal{K}_z(P_1) = -A_{z,P_1}\frac{d}{dz}\mathcal{K}_z(P_1), \tag{1.8}$$

$$P_1\gamma A_{z,P_2}^{-1}A_{z,P_1} = -\mathcal{K}_z(P_1)^{-1}\mathcal{K}_z(P_2)P_2\gamma = -S_z(P_1)S_z(P_2)^{-1}P_2\gamma, \tag{1.9}$$

$$P_2\frac{d}{dz}\gamma\mathcal{K}_z(P_1) = \frac{d}{dz}\left(\mathcal{K}_z(P_2)^{-1}\mathcal{K}_z(P_1)P_1\right) = \frac{d}{dz}\left(S_z(P_2)^{-1}\,S_z(P_1)P_1\right), \tag{1.10}$$

where $S_z := S_{A_z}$. Using (1.6) and (1.8) we obtain

$$\mathrm{Tr}\,(\dot{A}_z(A_{z,P_1}^{-1} - A_{z,P_2}^{-1})) = \mathrm{Tr}\,(P_1\gamma A_{z,P_2}^{-1}A_{z,P_1}\frac{d}{dz}\mathcal{K}_z(P_1)),$$

while (1.9), (1.10) reduce this to the right side of (1.7). □

We assume now that $E^1 = E^2$ and A is a first-order elliptic operator of Dirac type. Let $P_1, P_2 \in Gr_{-1}(A)$ with $P_1 - P_2$ smoothing. Assume further that A_{P_1}, A_{P_2} are invertible, ζ-admissible, with spectral cut R_θ. Then by setting $A_\lambda = A - \lambda$ for $\lambda \in \Gamma$, an application of Theorem 1 yields the following formula:

Theorem 2. *With the above assumptions (putting $S_A = S$)*

$$\frac{\det_{\zeta,\theta}(A_{P_1})}{\det_{\zeta,\theta}(A_{P_2})} = \det{}_F\left(\frac{S(P_1)}{P_1S(P_2)}\right)\cdot e^{-\mathrm{LIM}^{\theta}_{\lambda\to\infty}\log\det_F((P_1S_\lambda(P_2))^{-1}S_\lambda(P_1))}. \tag{1.11}$$

Further, the regularized limit is independent of the operator A, and depends only on the boundary conditions P_1, P_2.

The final statement follows from the fact that the left side of (1.7) is the logarithmic derivative of the left side of (1.11) ([2], Prop 1.3).

As an example, applied to *selfadjoint boundary problems* for a Dirac operator over an odd-dimensional spin manifold, equation (1.11) yields the relative determinant formula of [10] as a special case.

2. Geometry of the determinant line bundle

2.1. **Families of EBVPs.** Let $\pi : Z \longrightarrow B$ be a smooth Riemannian fibration of manifolds with fibre X_b diffeomorphic to a compact manifold X with boundary Y, and let $\mathcal{E}^i \to Z, i = 0, 1$, be vertical Clifford bundles with compatible connection. Then we have a family of Dirac operators $\mathbb{D} = \{D_b \mid b \in B\} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1$, with $\mathcal{F}^i$ the infinite-dimensional Fréchet bundle on B with fibre $C^\infty(X_b, E^i_b)$, where $E^i_b = \mathcal{E}^i_{|X_b}$. (Here $\mathbb{D}$ can be a family of total or 'chiral' Dirac operators.)

The corresponding structures are inherited on the boundary fibration $\partial\pi : \partial Z \longrightarrow B$ of closed manifolds with fibre $Y_b = \partial X_b$. We assume a product geometry in a collar neighborhood U of ∂Z so that in $U_b = [0, 1) \times \partial X_b$ one has $(D_b)_{|U_b} =$

$\sigma_b(\partial/\partial u + D_{Y_b})$, with $\mathbb{D}_{\partial Z} = \{D_{Y_b} \mid b \in B\} : \mathcal{F}_{\partial Z} \to \mathcal{F}_{\partial Z}$ the family of selfadjoint boundary Dirac operators defined by $\partial\pi$. Thus we have a smooth fibration of Grassmannians $Gr(\mathbb{D}) \to B$ with fibre $Gr_{-\infty}(D_b)$. A *Grassmann section* for $\mathbb{D}$ is defined to be a smooth section $\mathbb{P} = \{P_b \in Gr_{-\infty}(D_b) \mid b \in B\}$ of $Gr(\mathbb{D})$, any such section differing from the Calderón section $P(\mathbb{D}) = \{P(D_b) \mid b \in B\}$ by a smooth family of smoothing operators.

The primary role of a Grassmann section is to define a smooth family of EBVPs $(\mathbb{D}, \mathbb{P})$ parameterizing the operators $D_{P_b} = D_b : \mathrm{dom}\,(D_{P_b}) \longrightarrow L^2(X_b, E_b^1)$, and hence an associated determinant line bundle[2] $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ with determinant section $b \mapsto \det(D_{P_b})$. On the other hand, a Grassmann section defines a smooth infinite-rank Hermitian subbundle $\mathcal{W} \to B$ of $\mathcal{F}_{\partial Z}$ with fibre $W_b = \mathrm{range}\,(P_b) \subset L^2(Y_b, E^1_{b|Y})$. For each pair of Grassmann sections $\mathbb{P}_1, \mathbb{P}_2$ we therefore have a smooth family of boundary Fredholm operators

$$(\mathbb{P}_2, \mathbb{P}_1) = \{P_{1,b} \circ P_{2,b} : W_b^2 \to W_b^1\} \in C^\infty(B, \mathrm{Hom}(\mathcal{W}^2, \mathcal{W}^1)),$$

with determinant line bundle $\mathrm{DET}\,(\mathbb{P}_2, \mathbb{P}_1)$, and from [8] there is a canonical isomorphism

$$\mathrm{DET}\,(\mathbb{D}, \mathbb{P}_1) \cong \mathrm{DET}\,(\mathbb{D}, \mathbb{P}_2) \otimes \mathrm{DET}\,(\mathbb{P}_2, \mathbb{P}_1), \tag{2.1}$$

where $\mathbb{S}(\mathbb{P}_i) := (P(\mathbb{D}), \mathbb{P}_i)$. This means that the *abstract* relative determinant is a canonical section of the relative determinant line bundle $\mathrm{DET}\,(\mathbb{P}_2, \mathbb{P}_1)$. By choosing $\mathbb{P}_2 = P(\mathbb{D}), \mathbb{P}_1 = \mathbb{P}$ this becomes a section of

$$\mathrm{DET}\,(\mathbb{S}(\mathbb{P})) \cong \mathrm{DET}\,(\mathbb{D}, \mathbb{P}). \tag{2.2}$$

2.2. **The ζ and $\mathcal{C}$ metrics.** The bundle isomorphism (2.2) means that $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ inherits an Hermitian metric from $\mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$. This is the *canonical metric* $\|\cdot\|_{\mathcal{C}}$, given over the open subset $\Omega \subset B$ where the operators D_P are invertible by

$$\|\det D_P\|^2_{\mathcal{C}} := \det{}_F(S(P)^* S(P)) = \det{}_F(P(D) \cdot P \cdot P(D)).$$

Here $\Delta_P = D^* D : \mathrm{dom}\,(\Delta_P) \to L^2(X, E^0)$ is the Dirac Laplacian with domain

$$\mathrm{dom}\,(\Delta_P) = \{\psi \in H^2(X, E^0) : P\gamma\psi = 0, P^\star\gamma D\psi = 0\},$$

where $P^\star = \sigma(I - P)\sigma^{-1}$ is the adjoint boundary condition for D^* (thus $(D_P)^* = D^*_{P^\star}$).

On the other hand, from recent work of Grubb [3] we know that $\zeta(\Delta_P, s)$ is regular at $s = 0$ for $P \in Gr_{-\infty}(D)$. The resulting Quillen metric on $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ is defined over Ω by $\|\det D_P\|^2_\zeta = \det_\zeta \Delta_P$.

Theorem 3. *Let $\mathbb{P}_1$, $\mathbb{P}_2$ be Grassmann sections for $\mathbb{D}$ and let $D_{P_1} \in (\mathbb{D}, \mathbb{P}_1), D_{P_2} \in (\mathbb{D}, \mathbb{P}_2)$ be invertible at $b \in B$. Then*

$$\frac{\|\det(D_{P_1})\|_\zeta}{\|\det(D_{P_2})\|_\zeta} = \frac{\|\det(D_{P_1})\|_{\mathcal{C}}}{\|\det(D_{P_2})\|_{\mathcal{C}}}. \tag{2.3}$$

That is,

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \frac{\det_F(S(P_1)^* S(P_1))}{\det_F(S(P_2)^* S(P_2))}. \tag{2.4}$$

[2] For a smooth family of Fredholm operators $\mathbb{A} = \{A_b \mid b \in B\}$ parameterized by B, the complex lines $\mathrm{Det}\,(A_b) = \bigwedge^{\max} \mathrm{Ker}(A_b)^* \otimes \bigwedge^{\max} \mathrm{Coker}(A_b)$ fit together to define the determinant line bundle $\mathrm{DET}\,(\mathbb{A}) \to B$ endowed with a canonical section $b \mapsto \det(A_b)$ (see [1, 7, 8]).

Or, from (1.4),

$$\det_\zeta(\Delta_{P_1},\Delta_{P_2}) = \det_\zeta(S(P_1)^*S(P_1)\ ,\ S(P_2)^*S(P_2)). \tag{2.5}$$

Equivalently, since $S(P(D)) = Id$,

$$\det_\zeta(\Delta_P) = \det_\zeta(\Delta_{P(D)})\cdot\det_F(S(P)^*S(P))\ . \tag{2.6}$$

Remark 2.1. In fact, this holds for $P_1 - P_2$ differing just by a ψdo of order less than $-\dim(X)$. Equations (2.3)–(2.5) say that the *relative ζ-metric* and *relative $\mathcal{C}$-metric* on $\mathrm{DET}\,(\mathbb{D},\mathbb{P}_1)\otimes\mathrm{DET}\,(\mathbb{D},\mathbb{P}_2)^*$ coincide. Similarly, (2.6) corresponds to the isomorphism (2.2).

Proof. We study Δ_P by embedding in a fully elliptic first-order system. Associated to Δ we have the first-order elliptic operator acting on sections of $E^0\oplus E^1$

$$\widehat{\Delta} = \begin{pmatrix} 0 & D^* \\ D & -I\end{pmatrix} : H^1(X,E^0\oplus E^1)\to L^2(X,E^0\oplus E^1)\ .$$

In the collar $U_b\cong[0,1)\times Y$, $\widehat{\Delta}$ has the form $\widehat{\Delta}_{|U} = \widehat{\sigma}\left(\frac{\partial}{\partial u}+\widehat{B}+\widehat{R}\right)$, where

$$\widehat{\sigma} = \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0\end{pmatrix},\quad \widehat{B} = \begin{pmatrix} B & 0 \\ 0 & -\sigma B\sigma^{-1}\end{pmatrix},\quad \widehat{R} = \begin{pmatrix} 0 & -\sigma^{-1} \\ 0 & 0\end{pmatrix}\ ,$$

satisfying the relations $\widehat{\sigma}^2=-I$, $\widehat{\sigma}^*=-\widehat{\sigma}$, $\widehat{\sigma}\widehat{B}+\widehat{B}\widehat{\sigma}=0$, $\widehat{\sigma}\widehat{R}+\widehat{R}\widehat{\sigma}=-I$. Hence $\widehat{\Delta}$ is of Dirac type with Calderón projection $P(\widehat{\Delta})$ on $L^2(X,E^0_{|Y}\oplus E^1_{|Y})$ (see §2), and for each $\tilde{P}\in Gr_{-1}(\widehat{\Delta})$ we have a first-order EBVP

$$\widehat{\Delta}_{\tilde{P}} = \widehat{\Delta} : \mathrm{dom}\,(\widehat{\Delta}_{\tilde{P}})\to L^2(X,E^0\oplus E^1).$$

We recover the resolvent $(\Delta_P-\lambda)^{-1}$ via the canonical embeddings

$$Gr_{-1}(D)\longrightarrow Gr_{-1}(\widehat{\Delta}),\qquad P\longmapsto \widehat{P}:=P\oplus P^\star,$$

$$\widehat{\imath} : H^2(X,E^0)\longrightarrow H^2(X,E^0)\oplus H^1(X,E^1),\qquad \psi\longmapsto(\psi,D\psi)\ .$$

More precisely, setting $\widehat{\Delta}_\lambda = \left(\begin{smallmatrix} -\lambda & D^* \\ D & -I\end{smallmatrix}\right)$, $\widehat{\imath}$ restricts to an isomorphism $\mathrm{Ker}(\Delta-\lambda)\cong\mathrm{Ker}(\widehat{\Delta}_\lambda)$ and to an inclusion $\widehat{\imath}:\mathrm{dom}\,(\Delta_P)\to\mathrm{dom}\,(\widehat{\Delta}_{\lambda,\widehat{P}})$ into the domain of the first-order selfadjoint (note that $\widehat{\sigma}(I-\widehat{P})\widehat{\sigma}^{-1}=\widehat{P}$) local-elliptic boundary problem $\widehat{\Delta}_{\lambda,\widehat{P}}$, and one has

$$\widehat{\Delta}^{-1}_{\lambda,\widehat{P}} = \begin{pmatrix} (\Delta_P-\lambda)^{-1} & D^*_{P^\star}(\tilde{\Delta}_{P^\star}-\lambda)^{-1} \\ D_P(\Delta_P-\lambda)^{-1} & \lambda(\tilde{\Delta}_{P^\star}-\lambda)^{-1}\end{pmatrix}\ , \tag{2.7}$$

where $\tilde{\Delta}_{P^\star}=D_PD^*_{P^\star}$. The relative resolvent is trace class for $P_1,P_2\in Gr_{-\infty}(D)$, and we use (2.7) and (1.7) to compute that

$$\mathrm{Tr}\,\left((\Delta_{P_1}-\lambda)^{-1}-(\Delta_{P_2}-\lambda)^{-1}\right) = -\frac{\partial}{\partial\lambda}\log\det{}_F\left(\frac{S_\lambda(\widehat{P}_1)}{\widehat{P}_1S_\lambda(\widehat{P}_2)}\right),$$

where $S_\lambda(\widehat{P}):=S_{\widehat{\Delta}_\lambda}(\widehat{P})$. This determines the scattering operator for $(\Delta_{P_1},\Delta_{P_2})$. Combined with results from [3] we find that $\Delta_{P_1},\Delta_{P_2}$ are ζ-comparable. Applying Theorem 1 with $\theta=\pi$ we obtain

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \det{}_F\left(\frac{S(\widehat{P}_1)}{\widehat{P}_1S(\widehat{P}_2)}\right)\cdot e^{-\mathrm{LIM}_{\lambda\to+\infty}\log\det_F((\widehat{P}_1S_{-\lambda}(\widehat{P}_2))^{-1}S_{-\lambda}(\widehat{P}_1))}. \tag{2.8}$$

On the other hand, for a 1-parameter family $\Delta^r = D_r^* D_r$ the identity (1.7) leads to

$$\frac{d}{dr}\log\frac{\det_\zeta(\Delta^r_{P_1})}{\det_\zeta(\Delta^r_{P_2})} = \frac{d}{dr}\log\frac{\det_{\mathcal{C}}(\Delta^r_{P_1})}{\det_{\mathcal{C}}(\Delta^r_{P_2})}\ .$$

Combined with (2.8) we find that

$$\frac{\det_\zeta(\Delta_{P_1})}{\det_\zeta(\Delta_{P_2})} = \frac{\det_{\mathcal{C}}(\Delta_{P_1})}{\det_{\mathcal{C}}(\Delta_{P_2})}\cdot\alpha(P_1,P_2),$$

where $\alpha(P_1,P_2)$ depends only on P_1, P_2. Finally, by studying the 'vertical' boundary data variation

$$\frac{d}{dr}\log\det{}_\zeta(\Delta_{P_r}) = \mathrm{Tr}\,\left(S(\widehat{P}_r)^{-1}\frac{d}{dr}S(\widehat{P}_r)\right) - \underset{\lambda\to+\infty}{\mathrm{LIM}}\ \mathrm{Tr}\,\left(S_{-\lambda}(\widehat{P}_r)^{-1}\frac{d}{dr}S_{-\lambda}(\widehat{P}_r)\right),$$

the homogeneous (boundary symmetry group) structure of the Grassmannian forces $\alpha(P_1,P_2)=1$. (Note: It is not generally true that $\alpha=1$ for other classes of boundary conditions.) □

2.3. **The ζ- and $\mathcal{C}$-connection and curvature.** We briefly describe the extension of these methods to the regularized connection forms. To define a connection on $\mathrm{DET}\,(\mathbb{D},\mathbb{P})$ we make a choice of splitting $TZ = T(Z/B)\oplus T^H Z$. This induces a natural connection ∇^Z on $\mathcal{F}^i$ (see [1]). Similarly, we obtain a connection $\nabla^{\partial Z}$ on $\mathcal{F}_{\partial Z}$, and we assume that $\nabla^Z_{|U} = \gamma^*\nabla^{\partial Z}$. Given Grassmann sections $\mathbb{P}_i$, $i=1,2$, the associated subbundles $\mathcal{W}^i$ of $\mathcal{F}_{\partial Z}$ inherit a Hermitian metric with compatible connection $\nabla^i = \mathbb{P}_i\cdot\nabla^{\partial Z}\cdot\mathbb{P}_i$, and hence induce a connection $\nabla^{1,2}$ on $\mathrm{Hom}(\mathcal{W}^1,\mathcal{W}^2)$. The $\mathcal{C}$-connection on $\mathrm{DET}\,(\mathbb{D},\mathbb{P})$ is defined over Ω by $\nabla^{\mathcal{C}}\det(D_P)/\det(D_P) = \mathrm{Tr}\,(\mathcal{S}(P)^{-1}\nabla^{1,2}\mathcal{S}(P))$, with $\mathbb{P}_1 = P(\mathbb{D})$ and $\mathbb{P}_2=\mathbb{P}$. (See [8] for details.)

On the other hand, with a suitable boundary modification defined using $\mathbb{P}\cdot\nabla^{\partial Z}\cdot\mathbb{P}$, ∇^Z descends to a connection $\nabla^{\mathbb{P}}$ on the subbundle $\mathcal{F}_{\mathbb{P}}$ with fibre $\mathrm{dom}\,(D_{P_b})$ at $b\in B$. For $\Re(s) >> 0$, let $\omega_P(s) = -\mathrm{Tr}\,(\Delta_P^{-s}D_P\nabla^{\mathbb{P}}D_P^{-1})$. This has a continuation to $\mathbb{C}$ with a simple pole at $s=0$. Following [1], the ζ-connection form on $\mathrm{DET}\,(\mathbb{D},\mathbb{P})$ over Ω is defined by $(d/ds)_{|s=0}(s\omega(s))$. The $\mathcal{C}$- and ζ-connections are compatible with their metrics and we obtain the following result:

Theorem 4. *Let $\mathbb{P}_1$, $\mathbb{P}_2$ be choices of Grassmann sections. Let $\mathbf{R}_{\mathcal{C}}^{\mathbb{P}_i}, \mathbf{R}_\zeta^{\mathbb{P}_i}$ be the curvature 2-forms of the canonical and zeta connection on $\mathrm{DET}\,(\mathbb{D},\mathbb{P}_i)$. Then one has*

$$\mathbf{R}_\zeta^{\mathbb{P}_1} - \mathbf{R}_\zeta^{\mathbb{P}_2} = \mathbf{R}_{\mathcal{C}}^{\mathbb{P}_1} - \mathbf{R}_{\mathcal{C}}^{\mathbb{P}_2}. \tag{2.9}$$

Equivalently,

$$\mathbf{R}_\zeta^{\mathbb{P}} = \mathbf{R}_\zeta^{P(\mathbb{D})} + \mathbf{R}_{\mathcal{C}}^{\mathbb{P}}. \tag{2.10}$$

Equation (2.9) holds in $\Omega^2(B)$, since the endomorphism bundle of a complex line bundle is canonically trivial. The second identity (2.10), which says that the ζ curvature consists of an interior (cohomologically trivial) part plus a boundary correction term, follows from (2.9) because $\nabla^{\mathcal{C}}$ on $\mathrm{DET}\,(\mathbb{D},P(\mathbb{D}))$ is the trivial connection.

As an example, consider the 'universal' family $(\mathbb{D},\mathbb{P}) = \{D_P \mid P\in Gr_{-\infty}(D)\}$ of EBVPs parameterized by $Gr_{-\infty}(D)$ relative to a *fixed* operator D. Let $\mathbf{R}_\zeta$ be the ζ curvature of the corresponding determinant bundle. Then $\mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$ is the determinant bundle of [6], used to construct loop group representations, ∇^Y is just the trivial connection, $\mathbf{R}_\zeta^{P(\mathbb{D})}=0$, and $\nabla^{\mathcal{C}}$ the connection used in [6], and we obtain:

Corollary 2.2.

$$\mathbf{R}_\zeta = i\omega_{Gr}\,,$$

where ω_{Gr} is the Kähler form on the Grassmannian.

References

[1] Bismut, J. M., and Freed, D., 'The analysis of elliptic families: (I) Metrics and connections on determinant bundles', *Commun. Math. Phys.* **106** (1986), 159–176. MR **88h:**58110a
[2] Forman, R., 'Functional determinants and geometry', *Invent. Math.* **88** (1987), 447–493. MR **89b:**58212
[3] Grubb, G., 'Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems', *Ark. Mat.* **37** (1999), 45–86. MR **2000c:**35265
[4] Grubb, G., and Seeley, R., 'Zeta and eta functions for Atiyah-Patodi-Singer operators', *J. Geom. Anal.* **6** (1996), 31–77. MR **97g:**58174
[5] Lesch, M., and Tolksdorf, J., 'On the determinant of one-dimensional elliptic boundary value problems', *Comm. Math. Phys.* **193** (1998), 643–660. MR **2000b:**58060
[6] Pressley, A. and Segal, G. B., *Loop Groups*, Oxford, Clarendon Press, 1986. MR **88i:**22049
[7] Quillen, D. G., 'Determinants of Cauchy-Riemann operators over a Riemann surface', *Funktsional. Anal. i Prilozhen.* **19** (1985), no. 1, 37–41; English transl., *Functional Anal. Appl.* **19** (1985), 31–34. MR **86g:**32035
[8] Scott, S. G., 'Splitting the curvature of the determinant line bundle', *Proc. Amer. Math. Soc.* **128** (2000), 2763–2775. MR **2000m:**58055
[9] Scott, S. G., In preparation.
[10] Scott, S.G., and Wojciechowski, K.P., 'The ζ-determinant and Quillen's determinant for a Dirac operator on a manifold with boundary', *Geom. Funct. Anal.* **10** (2000), 1202–1236. CMP 2001:05

Department of Mathematics, King's College, London WC2R 2LS, U.K.
E-mail address: `sscott@mth.kcl.ac.uk`