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ON SPECTRA OF GEOMETRIC OPERATORS ON OPEN MANIFOLDS AND DIFFERENTIABLE GROUPOIDS

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ABSTRACT. We use a pseudodifferential calculus on differentiable groupoids to obtain new analytical results on geometric operators on certain noncompact Riemannian manifolds. The first step is to establish that the geometric operators belong to a pseudodifferential calculus on an associated differentiable groupoid. This then leads to Fredholmness criteria for geometric operators on suitable noncompact manifolds, as well as to an inductive procedure to compute their essential spectra. As an application, we answer a question of Melrose on the essential spectrum of the Laplace operator on manifolds with multicylindrical ends.

INTRODUCTION

On a compact manifold, the spectrum of an elliptic selfadjoint operator of positive order consists of isolated eigenvalues of finite multiplicities. In particular, the essential spectrum of such an operator is empty. These facts are part of the "elliptic package", which also includes boundedness, Fredholmness, and compactness criteria for (pseudo)differential operators acting between Sobolev spaces. It is well-known that the assumption of compactness of the underlying manifold is essential for some of these results.

For example, on a noncompact manifold with cylindrical ends the essential spectrum of the Laplace operator is nonempty. A manifold with multicylindrical ends is (locally at infinity) a product of manifolds with cylindrical ends. In [7] (Conjecture 7.1), Melrose conjectured that the spectrum of the Laplace operator on a manifold with multicylindrical ends has a precise form determined by Laplace operators on certain canonical lower-dimensional manifolds. In Theorem 6 we answer this question of Melrose in the affirmative. In a certain sense, one can view this theorem as part of an "elliptic package" for manifolds with multicylindrical ends.

Indeed, we show that there exists an elliptic package for a larger class of noncompact manifolds. The first part of this note is devoted to reviewing some results

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in this direction. We then use some of these results to answer Melrose's question. However, it should be pointed out that, at this point, understanding the fine structure of spectra requires some additional analysis that depends on more specific properties of the geometry of the noncompact manifold studied.

Our results apply to manifolds modeled by groupoids and use the pseudodifferential calculus on groupoids developed in [1, 4, 5, 8, 11] and [12]. We begin by quickly and informally reviewing some of the necessary definitions, including the definition of a differential groupoid $\mathcal{G}$ and that of the algebra $\Psi^\infty(\mathcal{G})$ of pseudodifferential operators on $\mathcal{G}$. The precise relation between the open manifold M_0 we study and groupoids is that M_0 has a compactification to a manifold with corners M, on which we are given a vector bundle A such that $A|_{M_0} \cong TM_0$, and $\Gamma(A)$ is naturally a Lie algebra with respect to the bracket induced by the Lie bracket of vector fields. (Thus, the vector bundle A is a Lie algebroid of a particular kind.) This is in the spirit of Melrose's approach to a pseudodifferential analysis on manifolds with corners and geometric scattering theory [7].

A metric on A as above restricts to a metric on TM_0, so that M_0 becomes, naturally, a Riemannian manifold. We are interested in studying the geometric operators associated to this metric (Dirac, Laplace, and so on). Our approach is along the lines of [7]. Thus, we first integrate A to a differential groupoid $\mathcal{G}$. This step has to be carried out in detail, as it is not true that every Lie algebroid is integrable. Then we check that the geometric operators on M_0 belong to our calculus $\Psi^\infty(\mathcal{G})$ (or $\Psi^\infty(\mathcal{G}; E)$, if they act on a vector bundle $E \to M$).

The restriction (an operation that has to be properly defined) of an operator in $\Psi^\infty(\mathcal{G})$ to a hyperface H of M belongs to the pseudodifferential calculus on the restricted groupoid $\mathcal{G}_H$. The restriction of a geometric operator to a hyperface is in fact also a geometric operator of the same kind. As expected from the work of Melrose on the b-calculus and related algebras, the results in the elliptic package for the action on M_0 of an operator $P \in \Psi^\infty(\mathcal{G}; E)$ are formulated not only in terms of the principal symbol of P, but also in terms of the restriction of P to the hyperfaces of M. For suitable $\mathcal{G}$, these results include:

- *Order zero operators are bounded on Sobolev spaces.*
- *An operator $P \in \Psi^\infty(\mathcal{G})$ is Fredholm between appropriate Sobolev spaces if, and only if, it is elliptic (i.e.,* its principal symbol is invertible) *and its restrictions to all hyperfaces are invertible as operators between Hilbert spaces.*
- *An order zero operator is compact if, and only if, its principal symbol and all its restrictions to hyperfaces vanish.*

Some of these results were proved before for certain classes of manifolds, most notably, for manifolds with multicylindrical ends by Melrose and Piazza [9]. An extensive list of references can be found in [4, 5, 6].

It is worth pointing out at this point that the well-known difficulties in connection with a parametric construction in the b-calculus were circumvented by studying the structure of the C^*-algebra closures of the operators of order 0, respectively, $-\infty$. In fact, the C^*-algebra setting is sufficient for a reasonable Fredholm theory for pseudodifferential operators of order 0. Operators of positive order, *e.g.* differential operators then can be treated using appropriate families of order reducing operators; for the spectral properties of selfadjoint elliptic operators of positive order, we use the Cayley transform to reduce to the case of norm limits of operators of order 0.

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1. Pseudodifferential operators on groupoids

We shall need groupoids endowed with various additional structures (see [2] and the references therein). Recall that a *groupoid* is a small category $\mathcal{G}$ all of whose morphisms are invertible. More precisely, a groupoid $\mathcal{G}$ is a pair $(\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ of sets together with structural morphisms d, r, μ, u, and ι. Here the first set, $\mathcal{G}^{(0)}$, represents the objects (or units) of the groupoid and the second set, $\mathcal{G}^{(1)}$, represents the set of morphisms of $\mathcal{G}$. Usually, we shall denote the space of units of $\mathcal{G}$ by M, and we shall identify $\mathcal{G}$ with $\mathcal{G}^{(1)}$. Each object of $\mathcal{G}$ can be identified with a morphism of $\mathcal{G}$, the identity morphism of that object, which leads to an injective map $u : M := \mathcal{G}^{(0)} \to \mathcal{G}$ used to identify M with a subset of $\mathcal{G}$. Each morphism $g \in \mathcal{G}$ has a "domain" and a "range", which are denoted by $d(g)$, the *domain*, and $r(g)$, the *range* of g, respectively. The multiplication (or composition) $\mu(g, h) = gh$ of two morphisms $g, h \in \mathcal{G}$ is only defined when $d(g) = r(h)$. The map $\iota : \mathcal{G} \to \mathcal{G} : g \mapsto g^{-1}$ is the "inverse" map. The structural maps are required to satisfy the usual compatibility relations satisfied by the morphisms of a category.

A *differentiable groupoid* is a groupoid $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, d, r, \mu, u, \iota)$ such that $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, the structural maps d, r, μ, u, and ι are differentiable, the domain map d is a submersion, and all the spaces $\mathcal{G}^{(0)}$ and $\mathcal{G}_x := d^{-1}(x)$, $x \in \mathcal{G}_0$, are Hausdorff.

The Lie algebroid $A(\mathcal{G})$ associated to a differentiable groupoid $\mathcal{G}$ is a bundle on $M = \mathcal{G}^{(0)}$ such that its smooth sections identify with the sections of the d-vertical tangent bundle to $\mathcal{G}$ that are invariant with respect to right translations on $\mathcal{G}$. Consequently, $\Gamma(A(\mathcal{G}))$ has a natural Lie algebra structure.

Definition 1 ([12])**.** A *pseudodifferential operator on* $\mathcal{G}$ is a uniformly supported family $P = (P_x)$ of pseudodifferential operators on the fibers $\mathcal{G}_x := d^{-1}(x)$, which form a smooth family invariant with respect to right translations by elements in $\mathcal{G}$.

This definition was first presented in July 1996 at the joint SIAM-AMS-MAA Meeting on Quantization in Mount Holyoke. A similar definition was independently announced in [10].

The support condition means that the support of $k_P(g) = K_{d(g)}(g, d(g))$ is compact in $\mathcal{G}$; here $K_x(g, g')$ denotes the Schwartz kernel of the operator P_x. The set of pseudodifferential operators of order m on $\mathcal{G}$ will be denoted by $\Psi^m(\mathcal{G})$. If $E \to M$ is a vector bundle, we can consider operators on $r^*(E)$, which leads to the algebra $\Psi^m(\mathcal{G}; E)$. Both $\Psi^\infty(\mathcal{G}) := \bigcup_m \Psi^m(\mathcal{G})$ and $\Psi^\infty(\mathcal{G}; E) := \bigcup_m \Psi^m(\mathcal{G}; E)$ are $*$-algebras that satisfy the usual symbolic properties of pseudodifferential operators, provided that one uses $A^*(\mathcal{G})$, the dual of the Lie algebroid of $\mathcal{G}$, instead of the cotangent bundle of M. This is the same idea as the one behind the introduction of various compressed cotangent bundles by Melrose. It is interesting to mention here that for the particular choice of the product groupoid $\mathcal{G} = M \times M$, we recover the usual pseudodifferential calculus on a smooth manifold M; for this choice of $\mathcal{G}$ we have $A(\mathcal{G}) = TM$. The principal symbol map $\sigma_m : \Psi^m(\mathcal{G}; E) \to \mathcal{C}^\infty(A^*(\mathcal{G}) \smallsetminus 0)$ is defined in analogy with the classical case, by defining it first for the operators on each of the spaces $\mathcal{G}_x$.

The algebra $\Psi^\infty(\mathcal{G})$ acts on $\mathcal{C}_c^\infty(M)$ and on each of the spaces $\mathcal{C}_c^\infty(\mathcal{G}_x)$. We denote these actions, or representations, by π and by π_x, respectively. The action on $\mathcal{G}_x$ is simply by restriction, and the action on M is determined by the relation $(\pi(P)f) \circ r = (Pf) \circ r$. If $P \in \Psi^0(\mathcal{G})$, then both $\pi(P)$ and $\pi_x(P)$ act as bounded operators. The representation π is called the *vector representation.*

2. Geometric operators

For two vector bundles E_0, E_1 on M, we shall denote by $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ the space of differential operators $D : \Gamma(\mathcal{G}; r^*E_0) \to \Gamma(\mathcal{G}; r^*E_1)$ with smooth coefficients that differentiate only along the fibers of $d : \mathcal{G} \to M$ and that are right invariant. Then, $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ is exactly the space of differential operators in $\Psi^m(\mathcal{G}; E_0, E_1)$. The elements of $\mathrm{Diff}(\mathcal{G}; E_0, E_1)$ will be called *differential operators on* $\mathcal{G}$. For example, the choice of a metric on $A(\mathcal{G})$ gives rise to a metric on each of the fibers $\mathcal{G}_x$ and hence to Hodge-Laplace operators acting on p-forms on these fibers; these operators form then a family $\Delta_p^{\mathcal{G}} \in \Psi^2(\mathcal{G}; \Lambda^p)$.

Suppose $Y \subset M$ is a submanifold such that $r^{-1}(Y) = d^{-1}(Y)$. Then Y is called an *invariant* subset, and $\mathcal{G}_Y := d^{-1}(Y)$ is also a differentiable groupoid, so we can consider the restriction of a family $P = (P_x) \in \Psi^m(\mathcal{G})$ to a family in $\Psi^m(\mathcal{G}_Y)$. The induced map $\Psi^m(\mathcal{G}) \to \Psi^m(\mathcal{G}_Y)$ will be denoted by $\mathcal{R}_Y$ and called the *restriction* (or *indicial*) morphism. For example, $\mathcal{R}_Y(\Delta_p^{\mathcal{G}}) = \Delta_p^{\mathcal{G}_Y}$.

Consider a bounded, nondegenerate representation $\varrho : \Psi^{-\infty}(\mathcal{G}) \longrightarrow \mathrm{End}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Then ϱ extends to a representation of $\Psi^\infty(\mathcal{G})$ by densely defined operators [4, 5].

Proposition 1. *Let $m > 0$, and let $A = A^* \in \Psi^m(\mathcal{G})$ be elliptic. Then the unbounded operator $\varrho(A) : \varrho(\Psi^{-\infty}(\mathcal{G}))\mathcal{H} \to \mathcal{H}$ is densely defined and essentially self-adjoint.*

Let us denote by $\mathfrak{A}(\mathcal{G})$ the closure of $\Psi^0(\mathcal{G})$ in the maximal (enveloping) C^*-norm $\|\cdot\|$ and by $C^*(\mathcal{G})$ the closure of $\Psi^{-\infty}(\mathcal{G})$ in the same norm. The norm $\|\cdot\|_r := \sup\|\pi_x(\,\cdot\,)\|$ is called the reduced norm. The following theorem allows us to define Sobolev spaces $H^s(\mathcal{H}, \varrho)$ associated to a representation ϱ of $\Psi^\infty(\mathcal{G})$ as the domains (or duals of the domains) of a fixed positive element. As a rule, when we consider algebras of operators acting on sections of a vector bundle E, we replace "$\mathcal{G}$" by "$\mathcal{G}; E$" in the notation.

Theorem 1. *Fix a differentiable groupoid $\mathcal{G}$ whose space of units, M, is compact. Let $D \in \Psi^m(\mathcal{G})$, $m > 0$, be such that $D \geq 1$ and $\sigma_m(D) > 0$. Then $D^{-s} \in C^*(\mathcal{G})$, for all $s > 0$. Moreover, if P has order $\leq k$, then $PD^{-k/m} \in \mathfrak{A}(\mathcal{G})$.*

The following corollary on the Cayley transform is especially useful.

Corollary 1. *If $A = A^* \in \Psi^m(\mathcal{G})$, $m > 0$, is elliptic, then the Cayley transform $(A + i)(A - i)^{-1}$ of A belongs to $\mathfrak{A}(\mathcal{G})$.*

The proofs of the above results depend essentially on the results of Landsman and Ramazan [3].

The above corollary is an important technical tool to handle parametrices (or more precisely the lack of parametrices) in certain pseudodifferential calculi, like the b-calculus. Also, it shows that while the inverse of an elliptic b-pseudodifferential operator may not be a b-pseudodifferential operator (in the so-called small calculus) anymore, it is nevertheless a uniform limit of such operators.

Assumption 1. From now on we shall assume that M_0 is a smooth manifold without corners which is diffeomorphic to (and will be identified with) an open dense subset of a compact manifold with corners M, and $\mathcal{G}$ will be a differentiable groupoid with units M, such that M_0 is an invariant subset and $\mathcal{G}_{M_0} \cong M_0 \times M_0$.

Note that the last condition ensures that on the open part M_0 elements in $\Psi^\infty(\mathcal{G})$ coincide with usual pseudodifferential operators. The special structure of the groupoid, however, takes care of the behavior at infinity.

The choice of a metric on $A(\mathcal{G})$ leads to geometric operators on $\mathcal{G}$ and M, which correspond to each other under the representation π of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M)$. For instance, π maps the Laplace operator on $\mathcal{G}$ to the Laplace operator on M.

The next example fits into this setting if M_0 is taken to be the interior of the manifold with corners M. The proof of Melrose's conjecture will be obtained by applying the general results we explain below to the following example. (Actually, for $c_H = 1$ [respectively, $c_H = 2$] we essentially recover Melrose's b- [respectively, c]-calculus).

Example 1. *The "very small" c_n-calculus.* Let M be a compact manifold with corners, and associate to each hypersurface $H \subset M$ an integer $c_H \geq 1$. Choose also on M a metric h which can be written in the neighborhood of each point p in the interior of a face $F \subseteq M$ of codimension k as $h = h_F + (dx_1)^2 + \cdots + (dx_k)^2$, with $x_1, \dots, x_k$ being the defining functions of F and h_F being a metric on F.

On M we consider the vector fields X locally of the form $X = X_F + \sum_{j=1}^k x_j^{c_j} \partial_{x_j}$, with c_j being the integer associated to the hyperface $\{x_j = 0\}$ and X_F being the lift of a vector field on F. The set of all vector fields with these properties forms a Lie subalgebra of the algebra of all vector fields on M, which can be identified with the smooth sections of a vector bundle $A(M, c)$.

For the interior S of a boundary face F of codimension k, let $\mathcal{G}_S = S \times S \times \mathbb{R}^k$, and let $\mathcal{G} := \bigcup_F S \times S \times \mathbb{R}^k$ be the groupoid with the obvious induced structural maps. As a set, $\mathcal{G}$ does not depend on c.

Proposition 2. *There exists on $\mathcal{G}$ a smooth structure which makes it a differentiable groupoid with $A(\mathcal{G}) \cong A(M, c)$. This smooth structure is uniquely determined.*

3. Spectral properties

Our analysis of geometric operators on M_0 depends on the structure of the algebras $\mathfrak{A}(\mathcal{G})$ and $C^*(\mathcal{G})$. Let $\mathfrak{I} = C^*(\mathcal{G}_{M_0})$; then $\mathfrak{I}$ is isomorphic to $\mathcal{K}(L^2(M_0))$, the algebra of compact operators on $L^2(M_0) = L^2(M)$, the isomorphism being induced by the vector representation π, or by any of the representations π_x, $x \in M_0$, and the isometry $\mathcal{G}_x \simeq M_0$. For $x \notin M_0$, π_x descends to a representation of $Q(\mathcal{G}) := \mathfrak{A}(\mathcal{G})/\mathfrak{I}$. Some of the results of this section were also obtained by Monthubert in his thesis; see [4] and the definitions therein.

We denote by $\sigma(P)$ the spectrum of an element $P \in \mathfrak{A}(\mathcal{G})$ and by $\sigma_{Q(\mathcal{G})}(P)$ the spectrum of the image of P in $Q(\mathcal{G}) = \mathfrak{A}(\mathcal{G})/C^*(\mathcal{G}_{M_0})$. These definitions extend to elliptic, selfadjoint elements $P \in \Psi^m(\mathcal{G})$, $m > 0$, using the Cayley transform, as follows. Let $f(t) = (t+i)/(t-i)$ and $f(P) := (P+i)(P-i)^{-1} \in \mathfrak{A}(\mathcal{G})$ be its Cayley transform, which is defined by Corollary 1. We define then

$$\sigma(P) := f^{-1}(\sigma(f(P))), \quad \text{and} \quad \sigma_{Q(\mathcal{G})}(P) := f^{-1}(\sigma_{Q(\mathcal{G})}(f(P))).$$

The spectrum and essential spectrum of an element T acting as an unbounded operator on a Hilbert space will be denoted by $\sigma(T)$ and, respectively, by $\sigma_{ess}(T)$.

We shall formulate all results below for operators acting on vector bundles. Fix an elliptic operator $A \in \Psi^m(\mathcal{G}; E)$, $m > 0$; then for any $P \in \Psi^k(\mathcal{G}; E)$, we have $P_1 := P(1+A^*A)^{-k/2m} \in \mathfrak{A}(\mathcal{G}; E)$, by Theorem 1. If π is the vector representation of $\Psi^0(\mathcal{G}; E)$ on $L^2(M; E) = L^2(M_0; E)$, then the spaces $H^s(M; E) = H^s(L^2(M), \pi)$ are the usual Sobolev spaces associated to the manifold of bounded geometry M_0.

Theorem 2. *For $P \in \Psi^m(\mathcal{G}; E)$, let $P_1 := P(1 + A^*A)^{-m/2k} \in \mathfrak{A}(\mathcal{G}; E)$.*

(i) If $P \in \Psi^m(\mathcal{G}; E)$ is such that the image of P_1 in $Q(\mathcal{G}; E)$ is invertible, then $\pi(P)$ extends to a bounded Fredholm operator $H^m(M; E) \to L^2(M; E)$.

(ii) If P_1 maps to zero in $Q(\mathcal{G}; E)$, then $\pi(P) : H^m(M; E) \to L^2(M; E)$ is a compact operator.

In particular, if $P \in \Psi^0(\mathcal{G}; E)$ or $P \in \Psi^m(\mathcal{G}; E)$, $m > 0$, is selfadjoint and elliptic, then $\sigma(\pi(P)) \subseteq \sigma(P)$ and $\sigma_{ess}(\pi(P)) \subseteq \sigma_{Q(\mathcal{G};E)}(P)$.

Proof. The inverse of P_1 up to $\mathfrak{I}$ leads directly to an inverse up to compact operators of $\pi(P)$. As for (ii), the operator $\pi(P) : H^m(M; E) \to L^2(M; E)$ is the product of the bounded operator $\pi(1 + A^*A)^{m/2k} : H^m(M; E) \to L^2(M; E)$ and of the compact operator $\pi(P_1)$.

For $m = 0$, the result on the spectrum is a general property of C^*-algebras. In general, the case $m > 0$ can be reduced to $m = 0$ using the Cayley transform and Corollary 1. ☐

This result can be sharpened to a necessary and sufficient condition for Fredholmness, respectively for compactness.

Theorem 3. *Assume that the vector representation π is injective on $C^*(\mathcal{G})$, and let $P \in \Psi^m(\mathcal{G}; E)$ be arbitrary. If $P \in \Psi^0(\mathcal{G}; E)$ or $P \in \Psi^m(\mathcal{G}; E)$, $m > 0$, is self-adjoint and elliptic, then we have $\sigma(\pi(P)) = \sigma(P)$ and $\sigma_{ess}(\pi(P)) = \sigma_{Q(\mathcal{G};E)}(P)$. In particular:*

(i) If $\pi(P)$ defines a Fredholm operator $H^m(M; E) \to L^2(M; E)$, then the image of $P(1 + A^*A)^{-m/2k} \in \mathfrak{A}(\mathcal{G}; E)$ in $Q(\mathcal{G}; E)$ is invertible.

(ii) If $\pi(P) : H^m(M; E) \to L^2(M; E)$ is compact, then $P(1 + A^*A)^{-m/2k}$ vanishes in $Q(\mathcal{G}; E)$.

Proof. Let $P_0 = P \in \mathfrak{A}(\mathcal{G}; E)$ if $m = 0$, or $P_0 = f(P) \in \mathfrak{A}(\mathcal{G}; E)$ if $m > 0$. By the very definition, it then suffices to consider P_0. An injective morphism of C^*-algebras preserves the spectrum; thus, $\sigma(P_0) = \sigma(\pi(P_0))$. Let $\mathcal{B}$ [respectively, $\mathcal{K}$] be the algebra of bounded [respectively, compact] operators on $L^2(M; E)$. Since the map $\pi' : Q(\mathcal{G}; E) \to \mathcal{B}/\mathcal{K}$ induced by the vector representation π is injective as well, we obtain $\sigma_{Q(\mathcal{G};E)}(P_0) = \sigma_{ess}(\pi(P_0))$. ☐

Recall that a groupoid $\mathcal{G}$ is called *amenable* if, and only if, the enveloping C^*-norm $\|\cdot\|$ and the reduced norm $\|\cdot\|_r$ coincide.

Theorem 4. *Suppose the restriction of $\mathcal{G}$ to $M \setminus M_0$ is amenable, and the vector representation π is injective. Then,*

(i) $P : H^s(M; E) \to L^2(M; E)$ is Fredholm if, and only if, P is elliptic and $\pi_x(P) : H^s(\mathcal{G}_x, r^*E) \to L^2(\mathcal{G}_x, r^*E)$ is invertible, for any $x \notin M_0$.

(ii) $P : H^s(M; E) \to L^2(M; E)$ is compact if, and only if, its principal symbol vanishes, and $\pi_x(P) = 0$, for all $x \notin M_0$.

(iii) For $P \in \Psi^0(\mathcal{G}; E)$, we have

$$\sigma_{ess}(\pi(P)) = \bigcup_{x \notin M_0} \sigma(\pi_x(P)) \cup \bigcup_{\xi \in S^*\mathcal{G}} \operatorname{spec}(\sigma_0(P)(\xi)),$$

where $\mathrm{spec}(\sigma_0(P)(\xi))$ *denotes the spectrum of the linear map* $\sigma_0(P)(\xi) : E_x \to E_x$.

(iv) If $P \in \Psi^m(\mathcal{G}; E)$, $m > 0$, *is formally selfadjoint and elliptic, then we have* $\sigma_{ess}(\pi(P)) = \bigcup_{x \notin M_0} \sigma(\pi_x(P))$.

Proof. We can certainly assume $E = \mathbb{C}$. The assumption $\mathfrak{A}(\mathcal{G}) = \mathfrak{A}_r(\mathcal{G})$ implies $\mathfrak{A}(\mathcal{G})/\mathfrak{I} = \mathfrak{A}_r(\mathcal{G})/\mathfrak{I}$. Since the groupoid obtained by reducing $\mathcal{G}$ to $M \smallsetminus M_0$ is amenable, the representation $\varrho := \prod \pi_x$, $x \notin M_0$ is injective on $Q(\mathcal{G})$. This gives $\sigma_{Q(\mathcal{G})}(T) = \bigcup_x \sigma(\pi_x(T))$, $x \notin M_0$, for all $T \in \mathfrak{A}(\mathcal{G})$. Together with Theorem 3 this gives (iii). Part (iv) is proved similarly using the Cayley transform, and (i) [respectively, (ii)] follow from (iii) [respectively, (iv)]. ☐

Another explicit criterion is contained in the theorem below.

Theorem 5. *Suppose the vector representation* π *is injective and* $M \setminus M_0$ *can be written as a union* $\bigcup_{j=1}^r Z_j$ *of closed, invariant manifolds with corners* $Z_j \subset M$.

(i) Assume also that each $\mathcal{G}|_{Z_j}$ *is amenable. Let* $P \in \Psi^m(\mathcal{G}; E)$, *then* $P : H^s(M; E) \to L^2(M; E)$ *is Fredholm if, and only if, it is elliptic and* $\mathcal{R}_{Z_j}(P) : H^s(\mathcal{G}|_{Z_j}; E|_{Z_j}) \to L^2(\mathcal{G}|_{Z_j}; E|_{Z_j})$ *is invertible, for all* j.

(ii) Let $P \in \Psi^m(\mathcal{G}; E)$; *then* $P : H^s(M; E) \to L^2(M; E)$ *is compact if, and only if, its principal symbol vanishes and* $\mathcal{R}_{Z_j}(P) = 0$, *for all* j.

(iii) For $P \in \Psi^0(\mathcal{G}; E)$, *we have*

$$\sigma_{ess}(\pi(P)) = \bigcup_{j=1}^r \sigma(\mathcal{R}_{Z_j}(P)) \cup \bigcup_{\xi \in S^*\mathcal{G}} \mathrm{spec}(\sigma_0(P)(\xi)) \,.$$

(iv) If $P \in \Psi^m(\mathcal{G}; E)$, $m > 0$, *is formally selfadjoint and elliptic, then we have* $\sigma_{ess}(\pi(P)) = \bigcup_{j=1}^r \sigma(\mathcal{R}_{Z_j}(P))$.

Proof. We assume that $E = \mathbb{C}$. The morphism

$$\mathfrak{A}(\mathcal{G})/\mathfrak{I} \to \bigoplus_j \mathfrak{A}(\mathcal{G}_{Z_j}) \oplus \mathcal{C}(S^*\mathcal{G}; \mathrm{End}(E))$$

given by the restrictions $\mathcal{R}_{Z_j}$ and the homogeneous principal symbol is injective. This gives (iii) and (iv). For $m > 0$ note that we have $\sigma_0(f(P)) = 1$ for the Cayley transform $f(P) = (P + i)(P - i)^{-1} \in \mathfrak{A}(\mathcal{G})$ of P, and $f^{-1}(1) = \{\infty\}$.

To obtain (i) and (ii) from (iii) as above, it is enough to observe that the operator $P_1 = P(1 + A^*A)^{-m/2k}$ (with A elliptic of order k, fixed) belongs to $\mathfrak{I} = C^*(\mathcal{G}_{M_0})$ if, and only if, $\sigma_0(P_1) = 0$ and $\mathcal{R}_{Z_j}(P_1) = 0$ for all j. Moreover, $\mathcal{R}_{Z_j}(P_1) = 0$ if, and only if, $\mathcal{R}_{Z_j}(P) = 0$. ☐

We are now going to apply the results of this section to the groupoid $\mathcal{G}(M, c)$ of Example 1. The main result is an inductive method for the determination of the essential spectrum of Hodge-Laplace operators. Because b-differential operators on M correspond to the differential operators in $\Psi^\infty(\mathcal{G}(M, c))$ if $c_H = 1$ for all boundary hyperfaces H of M, we in particular answer a question of Melrose on the essential spectrum of the b-Laplacian on the interior of a compact manifold M with corners [7, Conjecture 7.1] and natural complete metric (the interior of M_0 with this metric is sometimes also called a *manifold with multicylindrical ends*).

Lemma 1. *The groupoid* $\mathcal{G}(M, c)$ *is amenable, and the vector representation of* $\mathfrak{A}(\mathcal{G}(M, c))$ *is injective.*

Fix now a metric h on $A = A(\mathcal{G}(M,c))$, and let $\Delta_p^c := \Delta_p^{\mathcal{G}(M,c)}$ be the corresponding Hodge-Laplacian acting on p-forms.

Let us introduce some more notation. For each hyperface H of M, we consider the system $c^{(H)}$ determined by $c_F^{(H)} = c_{F'}$ for all boundary hyperfaces F' of M with F an open component of $H \cap F' \neq \emptyset$, as in Example 1. By the construction of $\mathcal{G}(M,c)$,

$$\mathcal{G}(M,c)_H \cong \mathcal{G}(H, c^{(H)}) \times \mathbb{R}.$$

It is convenient to use the Fourier transform to switch to the dual representation in the $\mathbb{R}$ variable, so that the action of the group by translation becomes an action by multiplication with the dual variable $\lambda \in \mathbb{R}^* \cong \mathbb{R}$. This reasoning then gives

$$\mathcal{R}_H(\Delta_p^c) = \Delta_p^{\mathcal{G}(M,c)_H} = \begin{cases} \lambda^2 + \Delta_0^{c^{(H)}} & \text{if } p = 0, \\ \left(\lambda^2 + \Delta_p^{c^{(H)}}\right) \oplus \left(\lambda^2 + \Delta_{p-1}^{c^{(H)}}\right) & \text{if } p > 0. \end{cases}$$

Denote $m_H^{(p)} = \min \sigma(\Delta_p^{c^{(H)}})$ and $m^{(p)} = \min_H m_H^{(p)}$. Then $m^{(p)} \geq 0$ because the Hodge-Laplace operators $\Delta_p^{c^{(H)}}$ are positive operators.

On the other hand, note that $\pi(\Delta_p^c)$ is equal to Δ_p, the Hodge-Laplace operator acting on p-forms on the complete manifold $M_0 := M \smallsetminus \partial M$, with the induced metric from $A(M,c)$.

Theorem 6. *Consider the open manifold M_0 which is the interior of a compact manifold with corners M, with the metric induced from $A(M,c)$. Then the essential spectrum of the Hodge-Laplacian Δ_p acting on p-forms on M_0 is $[m, \infty)$, with $m = m^{(0)}$, if $p = 0$, or $m = \min\{m^{(p)}, m^{(p-1)}\}$, if $p > 0$.*

Proof. The proof of the above theorem is obtained by applying Theorem 5 (iv), with Z_j ranging through the set of hyperfaces of M, using also Lemma 1. Then we notice that the restriction (or indicial) operators associated to these faces are the (family of) operators $\Delta_p(Z_j) + t^2$, where $\Delta_p(Z_j)$ are the Laplace operators on the interior of the hyperfaces Z_j and t is the variable dual to the variable normal to the hyperface Z_j. ☐

In particular, the spectrum of Δ_p itself is the union of $[m, \infty)$ and a discrete set consisting of eigenvalues of finite multiplicity. As pointed out to us by Melrose, $m^{(0)} = 0$ because M is compact, and hence every minimal face of M is also compact. Also, it is necessary to mention that our results say nothing about the more precise structure of the spectral decomposition of operators on manifolds with corners. It goes without saying, though, that these results are very interesting to pursue.

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