ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 7, Pages 87–93 (October 15, 2001)
S 1079-6762(01)00098-1

SOME NONEXISTENCE RESULTS FOR HIGHER-ORDER EVOLUTION INEQUALITIES IN CONE-LIKE DOMAINS

GENNADY G. LAPTEV

(Communicated by Guido Weiss)

ABSTRACT. Nonexistence of global (positive) solutions of semilinear higher-order evolution inequalities

$$\frac{\partial^k u}{\partial t^k} - \Delta u^m \geq |u|^q, \quad \frac{\partial^k u}{\partial t^k} - \Delta u \geq |x|^\sigma u^q, \quad \frac{\partial^k u}{\partial t^k} - \operatorname{div}(|x|^\alpha Du) \geq u^q$$

with $k = 1, 2, \ldots$, in cone-like domains is studied. The critical exponents q^* are found and the nonexistence results are proved for $1 < q \leq q^*$. Remark that the corresponding result for $k = 1$ (parabolic problem) is sharp.

1. Introduction

This paper is devoted to nonexistence of global (nontrivial) solutions for some semilinear evolution differential inequalities in unbounded cone-like domains. We can formulate the main goal of our investigation as follows: "There are many well-known results on nonexistence in cone-like domains for parabolic equations. Some results can be obtained also for hyperbolic problems. What are the common properties of the evolution equations that imply the nonexistence of global solutions?"

For our proof we use the test function method. We refer the interested readers to the book by Mitidieri and Pohozaev [14] and the references therein, and the papers [13, 18, 17, 15, 7].

Details of the proof can be found in the preprint [11]. The corresponding elliptic and parabolic problems in a cone were investigated (using the test function method) in [8, 14] and [10]. Some types of higher-order evolution equations in a ball were considered in [9].

Let S^{N-1} be the unit sphere in $\mathbf{R}^N$, $N \geq 3$, and (r, ω) polar coordinates in $\mathbf{R}^N$. Let K_ω be a domain of S^{N-1} with smooth boundary ∂K_ω. We denote by K the cone

$$K = \{(r, \omega) : 0 < r < +\infty, \omega \in K_\omega\}.$$

The lateral surface of the cone K is ∂K. "Cone-like domain" K_R, $R > 0$ denotes the set $\{x \in K : |x| > R\}$ with full surface ∂K_R.

Received by the editors April 7, 2001.
2000 *Mathematics Subject Classification.* Primary 35G25; Secondary 35R45, 35K55, 35L70.
The author was supported in part by RFBR Grant #01-01-00884.

Recall that the Laplace operator Δ in polar coordinates (r,ω) has the form

$$\Delta = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\omega = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega,$$

where Δ_ω denotes the Laplace–Beltrami operator on the unit sphere $S^{N-1}\subset \mathbf{R}^N$.

We shall use the first Helmholtz eigenvalue $\lambda_\omega \equiv \lambda_1(K_\omega) > 0$ and corresponding eigenfunction $\Phi(\omega)$ for the Dirichlet problem of Δ_ω in K_ω,

$$\begin{cases} \Delta_\omega\Phi + \lambda\Phi = 0 & \text{in } K_\omega, \\ \Phi|_{\partial K_\omega} = 0. \end{cases} \tag{1.1}$$

It is well known that $\Phi(\omega) > 0$ for $\omega \in K_\omega$. We assume $\Phi(\omega) \le 1$.

Let Ω be an unbounded domain in $\mathbf{R}^{N+1}$ with piecewise smooth boundary. We shall use the anisotropic Sobolev spaces $W_q^{2,k}(\Omega)$, and the local space $L_{q,\mathrm{loc}}(\Omega)$, whose elements belong to $L_q(\Omega')$ for any compact subset Ω': $\overline{\Omega'}\subset\Omega$. Denote the space of continuous functions by $C(\overline{\Omega})$.

The expression $\int_{\partial K}\frac{\partial u}{\partial n}\,dx$ denotes the integral of the directional derivative of u with respect to the outward normal n to the cone lateral surface ∂K.

2. Main results

Let $k\in\mathbf{N}$. Our model problem has the form

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \ge |u|^q & \text{in } K\times(0,\infty), \\ u|_{\partial K\times[0,\infty)} \ge 0, & \\ \frac{\partial^{k-1}u}{\partial t^{k-1}}|_{t=0} \ge 0, & u\not\equiv 0. \end{cases} \tag{2.1}$$

Definition 2.1. Let $u(x,t)\in C(\overline{K}\times[0,\infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i=1,\dots,k-1$, as $t=0$, are well defined. The function $u(x,t)$ is called a weak solution of (2.1) if for any nonnegative test function $\varphi(x,t)\in W_\infty^{2,k}(K\times(0,\infty))$ with compact support, such that $\varphi|_{\partial K\times(0,\infty)}=0$, the following inequality holds:

$$\begin{aligned}\int_0^\infty\int_{\partial K} u\frac{\partial\varphi}{\partial n}\,dx dt + \int_0^\infty\int_K u\left((-1)^k\frac{\partial^k\varphi}{\partial t^k} - \Delta\varphi\right)\,dxdt \\ \ge \int_0^\infty\int_K |u|^q\varphi\,dxdt + \sum_{i=1}^{k-1}(-1)^i\int_K \frac{\partial^{k-1-i}u}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\,dx \\ + \int_K\frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi(x,0)\,dx.\end{aligned}$$

Let us introduce the parameters

$$s^* = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\omega}, \qquad s_* = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\omega}; \tag{2.2}$$

here λ_ω is the first eigenvalue of the problem (1.1) introduced above. It is evident that $s^* - s_* = N-2$. These parameters go back to Kondrat'ev [6].

Theorem 2.2. *Let*

$$1 < q \le q_k^* = \frac{s^* + 2/k + 2}{s^* + 2/k} = 1 + \frac{2}{s^* + 2/k},$$

where s^* *is defined in* (2.2). *Then the problem* (2.1) *has no nontrivial global solution.*

The following theorem includes sharp results for a parabolic equation and inequality (i.e., $k = 1$):

Theorem 2.3. *Let*

$$1 < q \leq q_1^* = 1 + \frac{2}{s^* + 2},$$

where s^* *is defined in* (2.2). *Then the problem*

$$(2.3) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u \geq |u|^q & \text{in } K \times (0, \infty), \\ u|_{\partial K \times [0,\infty)} \geq 0, & \\ u|_{t=0} \geq 0, & u \not\equiv 0, \end{cases}$$

has no nontrivial global solution in the sense of Definition 2.1.

Results of this type were mentioned in the surveys [12] and [2]. Remark that we do not impose any additional constraints on the solution.

For $k = 2$ (hyperbolic problem) we have

Theorem 2.4. *Let*

$$1 < q \leq q_2^* = 1 + \frac{2}{s^* + 1},$$

where s^* *is defined in* (2.2). *Then the problem*

$$(2.4) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u \geq |u|^q & \text{in } K \times (0, \infty), \\ u|_{\partial K \times [0,\infty)} \geq 0, & \\ \frac{\partial u}{\partial t}|_{t=0} \geq 0, & u \not\equiv 0, \end{cases}$$

has no nontrivial global solution in the sense of Definition 2.1.

We do not know whether this result is sharp. But for hyperbolic problems in the whole $\mathbf{R}^N$ we can compare this critical exponent q_2^* with the classical Kato's result in $\mathbf{R}^N$. Namely, putting $s^* = N - 2$, we obtain Kato's exponent

$$q^* = \frac{N+1}{N-1},$$

but without any assumptions concerning the support of eventual solution. Such results for the general hyperbolic inequalities were obtained, among others, in [1, 4] (using Kato's technique) and [17] (with the test function approach). As shown in [17], this exponent is sharp in the weak sense.

It is interesting that formally passing to the limit as $k \to \infty$, we arrive at the sharp elliptic critical exponent (see, for example, [7, 8, 14] and the references therein)

$$q_\infty^* = 1 + \frac{2}{s^*}.$$

Let us consider the system

$$(2.5) \qquad \begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \geq |v|^{q_1} & \text{in } K \times (0, \infty), \\ \frac{\partial^k v}{\partial t^k} - \Delta v \geq |u|^{q_2} & \text{in } K \times (0, \infty), \\ u|_{\partial K \times [0,\infty)} \geq 0, \quad v|_{\partial K \times [0,\infty)} \geq 0, & \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}|_{t=0} \geq 0, \quad \frac{\partial^{k-1} v}{\partial t^{k-1}}|_{t=0} \geq 0, & u \not\equiv 0, \quad v \not\equiv 0. \end{cases}$$

Theorem 2.5. *Let $q_1 > 1$, $q_2 > 1$ and*

$$\max\{\gamma_1, \gamma_2\} \geq \frac{s^* + 2/k}{2}, \qquad \text{where} \quad \gamma_1 = \frac{q_1 + 1}{q_1 q_2 - 1}, \quad \gamma_2 = \frac{q_2 + 1}{q_1 q_2 - 1}.$$

Then (2.5) *has no nontrivial global solution.*

In this theorem the concept of solution is understood in the weak sense of Definition 2.1 (with some evident corrections; see for example [8, 9]).

For the parabolic system ($k = 1$)

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \geq |v|^{q_1} & \text{in } K \times (0, \infty), \\ \frac{\partial v}{\partial t} - \Delta v \geq |u|^{q_2} & \text{in } K \times (0, \infty), \\ u|_{\partial K \times [0,\infty)} \geq 0, \quad v|_{\partial K \times [0,\infty)} \geq 0, & \\ u|_{t=0} \geq 0, \quad v|_{t=0} \geq 0, & u \not\equiv 0, \quad v \not\equiv 0. \end{cases}$$

we obtain from Theorem 2.5 the well-known condition [12, 2]

$$\max\left\{\frac{q_1 + 1}{q_1 q_2 - 1}, \frac{q_2 + 1}{q_1 q_2 - 1}\right\} \geq \frac{s^* + 2}{2}.$$

Let us consider the inequality of porous medium type,

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u^m \geq |u|^q \quad \text{in } K \times (0, \infty), \qquad m \geq 1, \quad q > m, \\ u|_{\partial K \times [0,\infty)} \geq 0, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}\Big|_{t=0} \geq 0, \qquad u \not\equiv 0. \end{cases} \tag{2.6}$$

Definition 2.6. Let $u(x, t) \in C(\overline{K} \times [0, \infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i = 1, \ldots, k-1$, as $t = 0$, are well defined. The function $u(x, t)$ is called a weak solution of (2.6) if for any nonnegative test function $\varphi(x, t) \in W^{2,k}_\infty(K \times (0, \infty))$ with compact support, such that $\varphi|_{\partial K \times (0,\infty)} = 0$, the following inequality holds:

$$\begin{aligned} \int_0^\infty \int_{\partial K} u^m \frac{\partial \varphi}{\partial n} \, dx dt + (-1)^k \int_0^\infty \int_K u \frac{\partial^k \varphi}{\partial t^k} \, dx dt - \int_0^\infty \int_K u^m \Delta \varphi \, dx dt \\ \geq \int_0^\infty \int_K |u|^q \varphi \, dx dt + \sum_{i=1}^{k-1} (-1)^i \int_K \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x, 0) \frac{\partial^i \varphi}{\partial t^i}(x, 0) \, dx \\ + \int_K \frac{\partial^{k-1} u}{\partial t^{k-1}}(x, 0) \varphi(x, 0) \, dx. \end{aligned}$$

Theorem 2.7. *Let*

$$1 \leq m < q \leq q^* = \frac{m(s^* + 2) + 2/k}{s^* + 2/k} = m + 2\frac{m - m/k + 1/k}{s^* + 2/k},$$

where s^* is defined in (2.2)*. Then the problem* (2.6) *has no nontrivial global solution in the sense of Definition 2.6.*

In the "parabolic" case $k = 1$ we obtain

Theorem 2.8. *Let*

$$1 \leq m < q \leq q^* = m + \frac{2}{s^* + 2},$$

where s^* is defined in (2.2). Then the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u^m \geq |u|^q & \text{in } K \times (0,\infty), \qquad m \geq 1, \quad q > m, \\ u|_{\partial K \times [0,\infty)} \geq 0, \\ u|_{t=0} \geq 0, & u \not\equiv 0. \end{cases}$$

has no nontrivial global solution in the sense of Definition 2.6.

Now we turn our attention to the singular problem in cone-like domain K_R, $R > 0$,

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \operatorname{div}(|x|^\alpha Du) \geq u^q & \text{in } K_R \times (0,\infty), \\ u \geq 0, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}|_{t=0} \geq 0, & u \not\equiv 0, \end{cases} \tag{2.7}$$

where $-\infty < \alpha < 2$.

Remark that we deal with nonnegative solutions.

Definition 2.9. Let $u(x,t) \in C(\overline{K}_R \times [0,\infty))$ and suppose the locally summable traces $\frac{\partial^i u}{\partial t^i}$, $i = 1, \dots, k-1$, as $t = 0$, are well defined. The function $u(x,t)$ is called a weak solution of (2.7) if for any nonnegative test function $\varphi(x,t) \in W^{2,k}_\infty(K_R \times (0,\infty))$ with compact support, such that $\varphi|_{\partial K_R \times (0,\infty)} = 0$, the following inequality holds:

$$\begin{aligned} \int_0^\infty \int_{\partial K_R} u|x|^\alpha \frac{\partial \varphi}{\partial n}\,dx dt + \int_0^\infty \int_{K_R} u \left((-1)^k \frac{\partial^k \varphi}{\partial t^k} - \operatorname{div}(|x|^\alpha D\varphi) \right) dx dt \\ \geq \int_0^\infty \int_{K_R} u^q \varphi\,dx dt + \sum_{i=1}^{k-1} (-1)^i \int_{K_R} \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\,dx \\ + \int_{K_R} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi(x,0)\,dx. \end{aligned}$$

Let us introduce the parameters

$$s^*_\alpha = \frac{\alpha + N - 2}{2} + \sqrt{\left(\frac{\alpha + N - 2}{2} \right)^2 + \lambda_\omega},$$

$$s_{*\alpha} = -\frac{\alpha + N - 2}{2} + \sqrt{\left(\frac{\alpha + N - 2}{2} \right)^2 + \lambda_\omega}.$$

Theorem 2.10. *Let $-\infty < \alpha < 2$. If*

$$1 < q \leq q^* = \frac{s^*_\alpha + (2-\alpha)/k + 2 - \alpha}{s^*_\alpha + (2-\alpha)/k} = 1 + \frac{2-\alpha}{s^*_\alpha + (2-\alpha)/k},$$

then the problem (2.7) has no nontrivial nonnegative global solution.

In the parabolic case we have

Theorem 2.11. *Let $-\infty < \alpha < 2$. If*

$$1 < q \leq q^* = 1 + \frac{2-\alpha}{s^*_\alpha + 2 - \alpha},$$

then the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|x|^\alpha Du) \geq u^q & \text{in } K_R \times (0,\infty), \\ u \geq 0, & u \not\equiv 0, \end{cases}$$

has no nontrivial nonnegative global solution.

Acknowledgment

The author is grateful to Professor E. Mitidieri for setting up the problem and to Professor S. I. Pohozaev for helpful discussion of the results.

References

1. D. Del Santo, V. Georgiev and E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, In: *"Geometric Optics and Related Topics"* (Eds. F. Colombini & N. Lerner), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, pp. 117–140. Birkhäuser, Boston, 1997.
2. K. Deng and H. A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. **243** (2000), 85–126. MR **2001b:**35031
3. V. A. Galaktionov and H. A. Levine, *A general approach to critical Fujita exponents in nonlinear parabolic problems*, Nonlinear Anal. **34** (1998), 1005–1027. MR **99f:**35079
4. V. A. Galaktionov and S. I. Pohozaev, *Blow-up, critical exponents and asymptotic spectra for nonlinear hyperbolic equations*: Math. Preprint Univ. of Bath 00/10, 2000.
5. F. John, *Nonlinear wave equations, formation of singularities*, University Lecture Ser. **2**, Amer. Math. Soc., Providence, RI, 1990. MR **91g:**35001
6. V. A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical and angular points*, Trudy Moscov. Mat. Obshch. **16** (1967), 209–292. MR **37:**1777
7. V. V. Kurta, *On the absence of positive solutions to semilinear elliptic equations*, Tr. Mat. Inst. Steklova **227** (1999), 162–169; English transl., Proc. Steklov Inst. Math. **1999**, no. 4 (227), 155–162. MR **2001h:**35062
8. G. G. Laptev, *Absence of global positive solutions for systems of semilinear elliptic inequalities in cone*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), 108–124. CMP 2001:09
9. G. G. Laptev, *On nonexistence for a class of singular semilinear differential inequalities*, Tr. Mat. Inst. Steklova **232** (2001), 223–235.
10. G. G. Laptev, *Nonexistence results for semilinear parabolic differential inequalities in cone*, Mat. Sb., to appear.
11. G. G. Laptev, *Nonexistence of global solutions for higher-order evolution inequalities in unbounded cone-like domains*, preprint.
12. H. A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. **32** (1990), 262–288. MR **91j:**35135
13. E. Mitidieri and S. I. Pohozaev, *Absence of positive solutions for quasilinear elliptic problems on* $\mathbf{R}^N$, Tr. Mat. Inst. Steklova **227** (1999), 192–222; English transl., Proc. Steklov Inst. Math. **1999**, no. 4 (227), 186–216. MR **2001g:**35082
14. E. Mitidieri and S. I. Pohozaev, *A priori estimates and nonexistence of solutions to nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova **234** (2001).
15. S. I. Pohozaev and A. Tesei, *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), 99–109. MR **2001k:**35306
16. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A.P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Nauka, Moscow, 1987; English transl., Walter de Gruyter, Berlin/New York, 1995. MR **89a:**35002; MR **96b:**35003
17. L. Veron and S. I. Pohozaev, *Blow-up results for nonlinear hyperbolic inequalities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4). **29** (2000), 393–420.

18. Qi Zhang, *Blow-up results for nonlinear parabolic equations on manifolds*, Duke Math. J. **97** (1999), 515–539. MR **2000d:**35107
19. Qi Zhang, *Blow up and global existence of solutions to an inhomogeneous parabolic system*, J. Differential Equations **147** (1998), 155–183. MR **99d:**35073

Department of Function Theory, Steklov Mathematical Institute, Gubkina Street 8, Moscow, Russia
E-mail address: laptev@home.tula.net