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NONRATIONAL, NONSIMPLE CONVEX POLYTOPES IN SYMPLECTIC GEOMETRY

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Abstract. In this research announcement we associate to each convex polytope, possibly nonrational and nonsimple, a family of compact spaces that are stratified by quasifolds, i.e., the strata are locally modeled by $\mathbb{R}^k$ modulo the action of a discrete, possibly infinite, group. Each stratified space is endowed with a symplectic structure and a moment mapping having the property that its image gives the original polytope back. These spaces may be viewed as a natural generalization of symplectic toric varieties to the nonrational setting. We provide here the explicit construction of these spaces, and a thorough description of the stratification.

Introduction

To each rational convex polytope it is possible to associate, by a standard construction, a geometric object that is known as the toric variety corresponding to the polytope. Is it possible to associate a similar geometric object to a convex polytope that is not rational?

In a number of recent papers this object has been referred to as "nonexisting" (e.g., [BL] and, along the same lines, [BBFK1], [BBFK2]), but in fact it is shown by the authors in [P], [BP1] that for convex polytopes that are simple such an object (and in fact a whole family of such objects) exists; it is an example of a space, known as a quasifold, that is locally modeled by $\mathbb{R}^k$ modulo the action of a discrete, possibly infinite, group and represents a natural generalization of a toric variety.

In the present announcement we consider the problem from the symplectic viewpoint, in the case of convex polytopes that are no longer simple.

We show that given an n-dimensional vector space, $\mathfrak{d}$, and a convex polytope $\Delta \subset \mathfrak{d}^*$, there is a family of compact spaces that are stratified by symplectic quasifolds. Each space M of the family admits the continuous action of an n-dimensional quasitorus D and a continuous mapping $\Phi : M \longrightarrow \mathfrak{d}^*$ such that $\Phi(M) = \Delta$. The restriction of the D-action to each stratum is smooth and Hamiltonian, with

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moment mapping given by the restriction of Φ. These stratified spaces are a natural generalization to the quasifold setting of the notion of stratified symplectic space given in [LS].

The results announced in this paper are contained in [BP2].

In a subsequent paper, we will consider these spaces (as we have already done in the simple case) in the complex setting and therefore view them as a natural extension of the notion of toric variety (cf. Remark 3.11).

For the definition of symplectic quasifold, quasitorus and every related notion we refer the reader to [P].

1. Stratifications by quasifolds

We define the notion of space stratified by quasifolds in the generality we need for our purposes. For the general definition of stratification see [GM1], [GM2].

Definition 1.1. Let M be a compact topological space. A *decomposition of M by quasifolds* is a collection of disjoint locally closed connected quasifolds $\mathcal{T}_F$ ($F \in \mathcal{F}$), called *pieces*, such that

(1) The set $\mathcal{F}$ is finite and partially ordered;
(2) $M = \bigcup_F \mathcal{T}_F$;
(3) $\mathcal{T}_F \cap \overline{\mathcal{T}}_{F'} \neq \emptyset$ iff $\mathcal{T}_F \subseteq \overline{\mathcal{T}}_{F'}$ iff $F \leq F'$.

We also require that $\mathcal{F}$ has a maximal element F and that the corresponding piece $\mathcal{T}_F$ is open and dense in M. We call this piece the regular piece; the other pieces are called singular. We will then say that M is an *n-dimensional compact space decomposed by quasifolds*, with n the dimension of the regular set.

Remark 1.2. A standard construction that is useful for the definition of stratification is that of a cone over a compact space L decomposed by quasifolds. We will call *cone over* L, denoted by $\mathring{C}(L)$, the space $[0,1) \times L/\sim$, where two points (t,l) and (t',l') in $[0,1) \times L$ are equivalent if and only if $t = t' = 0$. This space is itself a space decomposed by quasifolds: for example, when L is a compact quasifold, the space $\mathring{C}(L)$ decomposes into two pieces: one is the cone point, the other is given by the quasifold $(0,1) \times L$. In fact we shall be considering a slightly more complicated situation: let t be a point in a quasifold $\mathcal{T}$, B an open neighborhood of t and L a compact space decomposed by quasifolds. The decomposition of L induces a decomposition of the product $B \times \mathring{C}(L)$: to each piece $\mathcal{L}$ of L there corresponds the piece $B \times (0,1) \times \mathcal{L}$; to cover the whole of $B \times \mathring{C}(L)$ we add a minimal piece, lying in the closure of all other pieces, given by B times the cone point.

A stratification is a decomposition that is locally well behaved.

Definition 1.3. Let M be an n-dimensional compact space decomposed by quasifolds; the decomposition of M is said to be a *stratification by quasifolds* if each singular piece $\mathcal{T}$, called *stratum*, satisfies the following conditions:

(i) let r be the dimension of $\mathcal{T}$; for every point $t \in \mathcal{T}$ there exist an open neighborhood U of t in M, an open neighborhood B of t in $\mathcal{T}$, an $(n-r-1)$-dimensional compact space L decomposed by quasifolds, called the *link* of t, and a homeomorphism $h : B \times \mathring{C}(L) \longrightarrow U$ that preserves the decompositions and that takes each piece of $B \times \mathring{C}(L)$ diffeomorphically into the corresponding piece of U;

(ii) the decomposition of L satisfies condition (i).

The definition is recursive and, since the dimension of L decreases at each step, we end up, after a finite number of steps, with links that are compact quasifolds.

2. The construction

Let $\mathfrak{d}$ be a real vector space of dimension n, and let Δ be a convex polytope of dimension n in the dual space $\mathfrak{d}^*$. We want to associate to the polytope Δ a family of compact spaces that are suitably stratified by symplectic quasifolds. We construct these spaces as symplectic quotients, following the procedure which was first introduced by Delzant in [D]. Write the polytope as

$$\Delta = \bigcap_{j=1}^{d} \{ \mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geq \lambda_j \} \tag{1}$$

for some elements $X_1, \ldots, X_d$ in the vector space $\mathfrak{d}$ and some real numbers $\lambda_1, \ldots,$ λ_d. Let Q be a quasilattice in the space $\mathfrak{d}$ containing the elements X_j (for example, the one that is generated by these elements) and let $\{e_1, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$; consider the surjective linear mapping

$$\begin{array}{rccc} \pi : & \mathbb{R}^d & \longrightarrow & \mathfrak{d}, \\ & e_j & \longmapsto & X_j. \end{array}$$

Consider the n-dimensional quasitorus $\mathfrak{d}/Q$. The mapping π induces a group homomorphism,

$$\Pi : T^d = \mathbb{R}^d/\mathbb{Z}^d \longrightarrow \mathfrak{d}/Q. \tag{2}$$

We define N to be the kernel of the mapping Π. The mapping Π defines an isomorphism

$$T^d/N \longrightarrow \mathfrak{d}/Q. \tag{3}$$

We construct a moment mapping for the Hamiltonian action of N on $\mathbb{C}^d$. Consider the mapping $J(\underline{z}) = \sum_{j=1}^{d}(|z_j|^2 + \lambda_j)e_j^*$, where the λ_j's are given in (1) and are uniquely determined by our choice of normal vectors. The mapping J is a moment mapping for the standard action of T^d on $\mathbb{C}^d$. Consider now the subgroup $N \subset T^d$ and the corresponding inclusion of Lie algebras $\iota : \mathfrak{n} \to \mathbb{R}^d$. The mapping $\Psi : \mathbb{C}^d \to \mathfrak{n}^*$ given by $\Psi = \iota^* \circ J$ is a moment mapping for the induced action of N on $\mathbb{C}^d$. We want to prove that the quotient $M = \Psi^{-1}(0)/N$, endowed with the quotient topology, is a space stratified by quasifolds. Notice that, by (2), the group N is not necessarily closed in T^d; moreover to each $\mathfrak{d}$ there corresponds a whole family of quotients, given by all possible choices of normal vectors and of quasilattices Q containing these vectors.

In our general setting, in which the polytope can be nonsimple, the zero set $\Psi^{-1}(0)$ is not in general a smooth submanifold of $\mathbb{R}^{2d}$. Nonsimpleness of the polytope is responsible, like in the rational case, for the decomposition in strata of the quotient, whilst nonrationality produces the quasifold structure of the strata.

To define the decomposition of M in pieces we start by giving some further definitions on the polytope Δ.

Let us consider the open faces of Δ. They can be described as follows. For each such face F there exists a possibly empty subset $I_F \subset \{1, \ldots, d\}$ such that

$$F = \{ \mu \in \Delta \mid \langle \mu, X_j \rangle = \lambda_j \ \text{ if and only if } j \in I_F \}. \tag{4}$$

A partial order on the set of all faces of Δ is defined by setting $F \leq F'$ (we say F is contained in F') if $F \subseteq \overline{F'}$. The polytope Δ is the disjoint union of its faces. Let $r_F = \mathrm{card}(I_F)$; we have the following definitions:

Definition 2.1. A p-dimensional face F of the polytope is said to be *nonsimple* or *singular* if $r_F > n - p$.

Definition 2.2. A p-dimensional face F of the polytope is said to be *simple* or *regular* if $r_F = n - p$.

Proposition 2.3. *The n-dimensional quasitorus $D = \mathfrak{d}/Q$ acts continuously on the topological space $M = \Psi^{-1}(0)/N$. Moreover M is compact and a continuous mapping $\Phi: M \longrightarrow \mathfrak{d}^*$ is defined such that $\Phi(M) = \Delta$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathfrak{d}^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} (\mathfrak{n})^* \longrightarrow 0. \tag{5}$$

By (5) we have that the mapping $(\pi^*)^{-1} \circ J$ induces a well defined mapping on the quotient M; we call this mapping Φ. Moreover $\underline{z} \in \Psi^{-1}(0)$ if and only if

$$|z_j|^2 = \langle \Phi([\underline{z}]), X_j \rangle - \lambda_j, \qquad j = 1, \ldots, d. \tag{6}$$

This implies that $\Phi(M) = \Delta$. Moreover properness of J implies that M is compact.

Remark 2.4. From the proof of Proposition 2.3 we deduce that for any face F of Δ the set $\Phi^{-1}(F)$ is nonempty and is precisely given by $\Psi^{-1}(0) \cap \mathbb{C}^d_F/N$, where $\mathbb{C}^d_F = \{\, (z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_j = 0 \text{ iff } j \in I_F \,\}$ (for further detail cf. [G], [P]).

3. The stratification

We are now ready to define the decomposition of M: the pieces are given by $\mathcal{T}_F = \Phi^{-1}(F)$ with F singular face, and by the union, over all nonsingular faces F, of the sets $\Phi^{-1}(F)$. This union is the regular piece of the decomposition while the $\mathcal{T}_F$'s are the singular pieces. Let $\mathrm{Int}(\Delta)$ be the open face of Δ; then the regular set contains $\Phi^{-1}(\mathrm{Int}(\Delta))$. We will label the regular set by the index $\mathrm{Int}(\Delta)$ and call it in short $\mathcal{T}_\Delta$. Remark 2.4 allows us to characterize the pieces $\mathcal{T}_F$ in the standard way, by the isotropy group attached to each of them.

Remark 3.1. Let F be a p-dimensional face. Let $\mathfrak{s}^F = \{\, (y_1, \ldots, y_d) \mid y_j = 0 \text{ if } j \notin I_F \,\}$ and $S^F = \{\, (Y_1, \ldots, Y_d) \in T^d \mid Y_j = 1 \text{ if } j \notin I_F \,\}$. The torus S^F is the stabilizer of T^d at any point $(z_1, \ldots, z_d) \in \Psi^{-1}(0) \cap \mathbb{C}^d_F$ and $\mathfrak{s}^F$ is its Lie algebra. The stabilizer of N on $\Psi^{-1}(0) \cap \mathbb{C}^d_F$ is then the $(r_F - n + p)$-dimensional subgroup N^F of N given by $N \cap S^F$. Its Lie algebra, $\mathfrak{n}^F$, is given by $\mathfrak{n} \cap \mathfrak{s}^F$. Notice that the regular set $\Psi^{-1}(0)_\Delta$, given by the union, over all nonsingular faces, of the sets $\Psi^{-1}(0) \cap \mathbb{C}^d_F$, has discrete stabilizer.

Theorem 3.2. *The subset $\mathcal{T}_F$ of M corresponding to each p-dimensional singular face of Δ is a $2p$-dimensional quasifold. The subset $\mathcal{T}_\Delta$ is a $2n$-dimensional quasifold. These subsets give a decomposition by quasifolds of M.*

Remark 3.3. A singular face has at most dimension $n-2$; therefore a singular piece has at most dimension $2n-4$.

Remark 3.4. The decomposition of M is induced by the decomposition of $\Psi^{-1}(0)$ given by the manifolds $\Psi^{-1}(0)\cap\mathbb{C}^d_F$, with F singular, and the open subset $\Psi^{-1}(0)_\Delta$ of $\Psi^{-1}(0)$. The quasifold structure of each piece $\mathcal{T}_F$ is naturally induced by the smooth structure of $\Psi^{-1}(0)\cap\mathbb{C}^d_F$; the quasifold structure of $\mathcal{T}_\Delta$ is induced by the smooth structure of $\Psi^{-1}(0)_\Delta$.

Let $p:\ \Psi^{-1}(0)\longrightarrow M$ be the projection; we have the following

Theorem 3.5. *Each piece $\mathcal{T}_F$ ($\mathcal{T}_\Delta$) of the decomposition of M has a natural symplectic structure induced by the quotient procedure, that is, its pull-back via p coincides with the restriction of the standard symplectic form of $\mathbb{C}^d$ to the manifold $\Psi^{-1}(0)\cap\mathbb{C}^d_F$ ($\Psi^{-1}(0)_\Delta$).*

Theorem 3.6. *The restriction of the D-action and of the mapping Φ to each piece of the space M is smooth, the action of D is Hamiltonian and a moment mapping is given by the restriction of Φ.*

Now we need to prove that our decomposition has a good local behavior. Let t be a point in the singular $2p$-dimensional piece $\mathcal{T}_F$; we want to construct a link of t satisfying Definition 1.3. Let $\mathfrak{s}^F_{\mathbb{C}}=\mathfrak{s}^F+i\mathfrak{s}^F$ be the complexification of the Lie algebra $\mathfrak{s}^F$. The mapping J restricted to $\mathfrak{s}^F_{\mathbb{C}}$ gives rise to a moment mapping for the action of S^F on $\mathfrak{s}^F_{\mathbb{C}}$; we denote this mapping by $J_F:\ \mathfrak{s}^F_{\mathbb{C}}\longrightarrow\left(\mathfrak{s}^F\right)^*$. Consider now the Hamiltonian action of the (r_F-n+p)-dimensional group N^F on $\mathfrak{s}^F_{\mathbb{C}}$, induced by that of S^F: a moment mapping is then given by $\psi_F=\iota_F^*\circ J_F$, where $\iota_F:\ \mathfrak{n}^F\longrightarrow\mathfrak{s}^F$ is the inclusion map. In fact, using (4), it turns out that $\psi_F=\sum_{j\in I_F}|z_j|^2\iota_F^*(e_j)$, hence $\psi_F^{-1}(0)$ is a cone. We can now construct the link of t: to begin with take the sphere $\mathcal{S}_{F,\epsilon}$ in $\mathfrak{s}^F_{\mathbb{C}}$ of radius ϵ, centered in 0; we have that for any given $\epsilon>0$ the space $\psi_F^{-1}(0)\cap\mathcal{S}_{F,\epsilon}$ is nonempty and is acted on by the group N^F. Let us denote the quotient, $\left(\psi_F^{-1}(0)\cap\mathcal{S}_{F,\epsilon}\right)/N^F$, by $L_{F,\epsilon}$. We have the following

Lemma 3.7. *Let t be a point in the singular piece $\mathcal{T}_F$. Then we can choose suitable open neighborhoods B_t and U_t of t, in $\mathcal{T}_F$ and M, respectively, and an $\epsilon>0$, such that a decomposition preserving homeomorphism $h_F:\ B_t\times\mathring{C}(L_{F,\epsilon})\longrightarrow U_t$ is defined. The mapping h_F restricted to each piece is a quasifold diffeomorphism. Moreover $L_{F,\epsilon}$ is a $(2n-2p-1)$-dimensional compact space decomposed by quasifolds.*

Theorem 3.8. *Let F be a singular face of the convex polytope Δ and t be a point in $\mathcal{T}_F$. The compact space $L_{F,\epsilon}$ is a link of t.*

The proof of Lemma 3.7 is based on Theorem 3.2 and on the explicit construction of the homeomorphism h_F from $B_t\times\mathring{C}(L_{F,\epsilon})$ onto U_t. The proof of Theorem 3.8 also consists in exhibiting explicitly, at each step of the recursive definition of link, a link for the point in consideration together with the corresponding homeomorphism.

Remark 3.9. Theorem 3.8 proves that the decomposition of M is in fact a stratification; moreover, from Theorems 3.2, 3.5, we know that each piece of the stratification of M has the structure of a symplectic quasifold, naturally induced by that of $\mathbb{C}^d$. We will call M a *space stratified by symplectic quasifolds.*

Remark 3.10. In the light of Theorem 3.6, we can view the mapping Φ as a moment mapping for the action of the n-dimensional quasitorus D on the $2n$-dimensional compact space M stratified by symplectic quasifolds. By Proposition 2.3, the image $\Phi(M)$ of the moment mapping Φ is exactly the polytope Δ.

Remark 3.11. The remark above emphasizes the relationship between the space M and the polytope Δ, which is very neat in the symplectic setting. From the complex point of view we have a compact space X, homeomorphic to M, stratified by complex quasifolds; X is n-dimensional and is acted on by the complexified torus $D_{\mathbb{C}}$ of same dimension. Such an action has a dense open orbit, corresponding to the open set $\Phi^{-1}(\mathrm{Int}(\Delta))$.

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