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A C^0-THEORY FOR THE BLOW-UP OF SECOND ORDER ELLIPTIC EQUATIONS OF CRITICAL SOBOLEV GROWTH

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ABSTRACT. Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $\Delta_g = -div_g \nabla$ the Laplace-Beltrami operator. Also let $2^\star$ be the critical Sobolev exponent for the embedding of the Sobolev space $H_1^2(M)$ into Lebesgue spaces, and h a smooth function on M. Elliptic equations of critical Sobolev growth like

$$\Delta_g u + hu = u^{2^\star - 1}$$

have been the target of investigation for decades. A very nice H_1^2-theory for the asymptotic behaviour of solutions of such an equation is available since the 1980's. In this announcement we present the C^0-theory we have recently developed. Such a theory provides sharp pointwise estimates for the asymptotic behaviour of solutions of the above equation.

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We denote by $H_1^2(M)$ the standard Sobolev space of functions in L^2 with one derivative in L^2. We consider equations like

$$\Delta_g u + hu = u^{2^\star - 1}, \tag{1}$$

where $\Delta_g = -div_g \nabla$ is the Laplace-Beltrami operator, $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of the Sobolev space $H_1^2(M)$ into Lebesgue spaces, h is a $C^{0,\theta}$ function on M, $0 < \theta < 1$, and u is required to be positive. We regard (1) as a possible model for second order elliptic equations of critical Sobolev growth. We let (h_α) be a sequence of $C^{0,\theta}$ functions on M, $0 < \theta < 1$, and let (u_α) be a bounded sequence in $H_1^2(M)$ of solutions of (1) in the sense that for any α,

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1} \tag{2}$$

and $\|u_\alpha\|_{H_1^2} \leq \Lambda$, where $\Lambda > 0$ is independent of α. We also assume that the h_α's are uniformly bounded and that they converge in L^2 to some limiting function h_∞. Then, thanks to Struwe [22], we know how to describe the asymptotic behaviour of the u_α's as $\alpha \to +\infty$. More precisely, it follows from Struwe [22] that, up to a subsequence,

$$u_\alpha = u^0 + \sum B_\alpha^i + R_\alpha, \tag{3}$$

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where u^0 is a solution of the limit equation

$$\Delta_g u + h_\infty u = u^{2^\star - 1},$$

the sum on the right hand side of (3) is a finite sum over i, B_α^i is a bubble obtained by rescaling fundamental positive solutions of the Euclidean equation $\Delta u = u^{2^\star - 1}$, and the R_α's are lower order terms which converge strongly to 0 in $H_1^2(M)$. This asymptotic description provides a very satisfactory H_1^2-theory for the asymptotic behaviour of solutions of equations like (1). Let us assume now that the h_α's converge $C^{0,\theta}$ to h_∞ for some $0 < \theta < 1$. An important issue in the study of equations like (1) is to get a theory in which the above asymptotic description holds also in the C^0-space, where pointwise estimates are involved. Such a C^0-theory was developed in Druet, Hebey and Robert [9]. We present the theory in this announcement.

We know from the Euclidean Sobolev inequality that there exists $K > 0$ such that for any smooth function u with compact support in $\mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{1/2^\star} \leq K \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{1/2} .$$

The sharp constant K in this inequality is

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}},$$

where ω_n is the volume of the unit n-sphere. We let u be the function on $\mathbb{R}^n$ given by

$$u(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-\frac{n}{2}} .$$

It is easily seen that u is an extremal function for the sharp Euclidean Sobolev inequality. Thanks to Caffarelli, Gidas and Spruck [3], u is also the unique positive solution of the critical Euclidean equation

$$\Delta u = u^{2^\star - 1}$$

which is such that $u(0) = \max_{\mathbb{R}^n} u = 1$. Its energy $E(u) = \|u\|_{2^\star}$ is given by $E(u) = K_n^{-(n-2)/2}$. From now on, we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We also let (x_α) be a converging sequence of points in M, and (μ_α) a sequence of positive real numbers converging to 0 as $\alpha \to +\infty$. We define the standard bubble with respect to the x_α's and μ_α's as the sequence (B_α) of functions on M given by

$$B_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha, x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}},$$

where d_g is the distance with respect to g. In other words, standard bubbles are rescalings of fundamental solutions of the critical Euclidean equation $\Delta u = u^{2^\star - 1}$. Now we return to equation (2), and we assume that there exist $0 < \theta < 1$ and a $C^{0,\theta}$ function h_∞ on M such that

$$\begin{aligned} &\text{the operator } \Delta_g + h_\infty \text{ is coercive , and} \\ &h_\alpha \to h_\infty \text{ in } C^{0,\theta}(M) \text{ as } \alpha \to +\infty . \end{aligned} \tag{4}$$

We let (u_α) be a sequence of solutions of (2), $u_\alpha \in C^{2,\theta}(M)$, u_α positive. The existence of u_α implies the coercivity of the operator $\Delta_g + h_\alpha$. An estimate like the one in Theorem 1 below implies the coercivity of the operator $\Delta_g + h_\infty$ when $N \geq 1$. We also assume that there exists $\Lambda > 0$ such that $E(u_\alpha) \leq \Lambda$ for all α, where $E(u) = \|u\|_{2^\star}$. Multiplying (2) by u_α and integrating over M, it is easily seen that the u_α's are bounded in $H_1^2(M)$. Up to a subsequence, we may therefore assume that for some $u^0 \in H_1^2(M)$,

$$u_\alpha \rightharpoonup u^0 \text{ weakly in } H_1^2(M) \tag{5}$$

as $\alpha \to +\infty$. Then u^0 is a solution of the limit equation

$$\Delta_g u + h_\infty u = u^{2^\star - 1} . \tag{6}$$

By the maximum principle and regularity theory, $u^0 \in C^{2,\theta}(M)$ and either $u^0 \equiv 0$ or $u^0 > 0$ everywhere in M. As already mentioned, we know from Struwe [22] that the u_α's express as u^0, plus a sum of standard bubbles, plus lower order terms in $H_1^2(M)$. We want to prove that the u_α's are C^0-controlled, from below and from above, by u^0 and standard bubbles. Our theorem is as follows:

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, (h_α) a sequence of $C^{0,\theta}$ functions on M such that (4) is satisfied, and (u_α) a sequence of positive solutions to (2) such that $E(u_\alpha) \leq \Lambda$ for some $\Lambda > 0$ and all α. Then there exist $N \in \mathbb{N}$, converging sequences $(x_{i,\alpha})$ in M, and sequences $(\mu_{i,\alpha})$ of positive real numbers converging to 0, $i = 1, \ldots, N$, such that, up to a subsequence,*

$$\begin{aligned} &(1-\varepsilon_\alpha)\, u^0(x) + \frac{1}{C}\sum_{i=1}^N \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \\ &\qquad \leq u_\alpha(x) \leq (1+\varepsilon_\alpha)\, u^0(x) + C\sum_{i=1}^N \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \end{aligned}$$

for all $x \in M$ and all α, where u^0 is the solution of the limit equation (6) given by (5), $C > 1$ is independent of α and x, and (ε_α), independent of x, is a sequence of positive real numbers converging to 0 as $\alpha \to +\infty$. In particular, the u_α's are C^0-controlled, on both sides, by u^0 and standard bubbles.

A complement to Theorem 1 is that C can be chosen as close as we want to 1 if we restrict the equation in Theorem 1 to small neighbourhoods of the geometrical blow-up points, defined as the limits of the $x_{i,\alpha}$'s. For instance, if $u^0 \not\equiv 0$, or if the u_α's just have one geometrical blow-up point, then for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, up to a subsequence,

$$\begin{aligned} &(1-\varepsilon_\alpha)\, u^0(x) + \frac{1}{1+\varepsilon}\sum_{i=1}^N \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \\ &\qquad \leq u_\alpha(x) \leq (1+\varepsilon_\alpha)\, u^0(x) + (1+\varepsilon)\sum_{i=1}^N \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \end{aligned}$$

for all α, all $x_0 \in \mathcal{S}$, and all $x \in B_{x_0}(\delta_\varepsilon)$, where $\mathcal{S}$ is the set consisting of the limits of the $x_{i,\alpha}$'s as $\alpha \to +\infty$, and $B_{x_0}(\delta_\varepsilon)$ is the geodesic ball of center x_0 and radius

δ_ε. Outside the $B_x(\delta_\varepsilon)$'s, $x \in \mathcal{S}$, the u_α's converge $C^{2,\theta}$ to u^0. The estimate then extends to M in the particular case where $u^0 \not\equiv 0$. A refined estimate on the u_α's is given below in (7). Another complement to Theorem 1 is that the bubbles in this theorem satisfy the Struwe H_1^2-decomposition [22]. More precisely, we also have that for any $x \in M$ and any α,

$$u_\alpha(x) = u^0(x) + \sum_{i=1}^N \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}} \right)^{\frac{n-2}{2}} + R_\alpha(x),$$

where the R_α's, $R_\alpha \in H_1^2(M)$ for all α, are such that $R_\alpha \to 0$ strongly in $H_1^2(M)$ as $\alpha \to +\infty$. Moreover,

$$E(u_\alpha)^{2^\star} = E(u^0)^{2^\star} + NK_n^{-n} + r_\alpha$$

for all α, where the r_α's, $r_\alpha \in \mathbb{R}$, are such that $r_\alpha \to 0$ as $\alpha \to +\infty$. In other words, the energies split also.

Such a theorem has important applications when dealing with sharp Sobolev inequalities. In this case $N = 1$, and the estimate in Theorem 1 appeared to be a key point when discussing the validity and the existence of extremal functions for sharp inequalities like

$$\|u\|_{2^\star}^2 \le K_n^2 \|\nabla u\|_2^2 + B\|u\|_2^2,$$

where B is a constant independent of u. Monographs on sharp constant problems are Druet-Hebey [7], and Hebey [12]. Other directions of research are dealing with the energy function (see for instance Hebey [13]) or with compactness results as in Schoen [19, 20]. Applications of the theorem in such directions are in Druet [6], where compactness results and low dimension phenomena are discussed, and the lower semicontinuity of the energy function for equations like (2) is proved. The energy function for equations like (2) is defined by

$$E(\alpha) = \inf_{u \in \mathcal{S}_\alpha} E(u),$$

where $E(u) = \|u\|_{2^\star}$ and $\mathcal{S}_\alpha$ is the set consisting of the solutions of (2). From the historical viewpoint, without any pretention to exhaustivity, Atkinson-Peletier [1] and Brezis-Peletier [2] have been concerned with the description of the pointwise behaviour of sequences of solutions of equations like (1), dealing with radially symmetrical solutions u_ε of the semicritical equations $\Delta u = u^{2^\star-1-\varepsilon}$ on the unit ball of the Euclidean space. More recent developments in this specific direction are in Robert [17, 18]. An estimate like in Theorem 1 in the case $N = 1$, stating that solutions of minimal energy of equations like (1) are controlled from above by a standard bubble, appeared then in Han [11] when dealing with solutions u_ε of the equations $\Delta u = u^{2^\star-1-\varepsilon}$ on bounded open subsets of the Euclidean space, in Hebey-Vaugon [14] when dealing with (1) and arbitrary Riemannian manifolds, and in Li [15, 16] when dealing with equations like (1) on the unit sphere. Improvements, still in the case $N = 1$, are in Druet [4, 5] and Druet-Robert [10]. We refer also to Schoen-Zhang [21]. Examples of blowing-up sequences of solutions of equations like (1) are in Druet-Hebey [8] and Druet-Hebey-Robert [9].

1. Proof of the theorem

We only present a very brief sketch of the proof. Details on the special case where $N = 1$, which is actually due to Druet and Robert, can be found in the monograph

[7] by Druet and Hebey. The intricate general case where N is arbitrary is in Druet, Hebey, and Robert [9]. We assume in what follows that $\max u_\alpha \to +\infty$ as $\alpha \to +\infty$. If not the case, up to a subsequence, the u_α's converge C^2 to u^0, and the theorem is true. As a remark, the proof of the theorem goes through the proof of a slightly stronger result. All that follows is up to a subsequence. We let G be the Green's function of the operator $\Delta_g + h_\infty$, and let Φ be the continuous function on $M \times M$ given by

$$\Phi(x, y) = (n-2)\omega_{n-1} d_g(x, y)^{n-2} G(x, y)$$

if $x \neq y$, and $\Phi(x, y) = 1$ if $x = y$, where ω_{n-1} is the volume of the unit $(n-1)$-sphere. Then we claim that for any converging sequence (x_α) of points in M, and any α,

$$u_\alpha(x_\alpha) = \Big(1 + o(1)\Big) u^0(x_\alpha) + \sum_{i=1}^{N} \Big(\Phi(x_i, x) + o(1)\Big) B_\alpha^i(x_\alpha), \tag{7}$$

where x is the limit of the x_α's, x_i is the limit of the $x_{i,\alpha}$'s, and B_α^i is the standard bubble with respect to the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s given by Theorem 1. It is easily seen that the theorem and the remarks after this theorem follow from such an asymptotic description. The proof of these asymptotics splits into four main steps. The first preliminary step consists in getting rescaling invariant estimates. Such estimates basically state that there exist $N \in \mathbb{N}^\star$, converging sequences $(x_{i,\alpha})$ in M, and sequences $(\mu_{i,\alpha})$ of positive real numbers converging to 0, $i = 1, \ldots, N$, such that

$$R_\alpha^N(x)^{\frac{n}{2}-1} u_\alpha(x) \leq C$$

for all x and all α, where $R_\alpha^N(x)$ is the minimum over i of the distances from the $x_{i,\alpha}$'s to x. The key idea here is that if such an estimate is false, then we can construct another blow-up point. This weak estimate comes with an important refinement and complementary informations on the limit of the u_α's outside blow-up points. Then we need to prove that the upper estimate in Theorem 1 holds. For that purpose, we rearrange the $x_{i,\alpha}$'s in families. Inside a family, blow-up points are close to each other. Two families are far from each other. If the $y_{i,\alpha}$'s are the representatives of such families having the largest $\mu_{i,\alpha}$, $i = 1, \ldots, k$, the second step in the proof consists in proving that an upper estimate like in Theorem 1 holds with respect to the $y_{i,\alpha}$'s. We prove that there exist $C > 0$ and $R > 0$ such that

$$u_\alpha(x) \leq \big(1 + o(1)\big) u^0(x) + C \sum_{i=1}^{k} B_\alpha^i(x)$$

for all α and all $x \in M \backslash \bigcup_{i=1}^k B_{y_{i,\alpha}}(R\mu_{i,\alpha})$, where B_α^i is the standard bubble with respect to $y_{i,\alpha}$ and its corresponding $\mu_{i,\alpha}$. The proof of such an estimate goes through the establishment of a scale of intermediate estimates, referred to as ε-sharp estimates with $0 < \varepsilon < \frac{n-2}{2}$, the weakest of these when $\varepsilon = \frac{n-2}{2}$ being like the weak estimate we discussed above. The proof of the upper estimate as in the theorem then reduces to the proof that this estimate holds inside the $B_{y_{i,\alpha}}(R\mu_{i,\alpha})$'s. This is the third step in the proof. We proceed here by induction. We consider subfamilies of blow-up points, and prove that the estimate holds in $B_{y_{i,\alpha}}(R\mu_{i,\alpha})$, outside smaller balls of sub-representatives, and so on up to the point where we have exhausted all the blow-up points. In each step of this induction process, we pass through ε-sharp estimates, and, in some sense, let then $\varepsilon \to 0$ to get the sharp estimate. Once we have proved that the upper estimate of Theorem 1 holds in M, the argument becomes simpler and more conventional. The fourth step in the proof consists in proving that the above asymptotics follow from the Green representation formula

$$\begin{aligned} u_\alpha(x_\alpha) - u^0(x_\alpha) &= \int_M G_\alpha(x_\alpha, x)\big(u_\alpha(x)^{2^\star-1} - u^0(x)^{2^\star-1}\big)dv_g \\ &\quad + \int_M G_\alpha(x_\alpha, x)\big(h_\infty(x) - h_\alpha(x)\big)u^0(x)dv_g \end{aligned}$$

where G_α is the Green's function for the operator $\Delta_g + h_\alpha$. The different terms that are involved in this formula are controlled thanks to the upper estimate we have just discussed. The asymptotics follow from rather standard developments. As already mentioned, since Φ is continuous, the theorem and the remarks after the theorem are then easy consequences of the asymptotics. We refer to Druet, Hebey and Robert [9] for more details, and also to Druet and Hebey [7] for the special case where $N = 1$.

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