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MARKOV STRUCTURES FOR NON-UNIFORMLY EXPANDING MAPS ON COMPACT MANIFOLDS IN ARBITRARY DIMENSION

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Abstract. We consider non-uniformly expanding maps on compact Riemannian manifolds of arbitrary dimension, possibly having discontinuities and/or critical sets, and show that under some general conditions they admit an induced Markov tower structure for which the decay of the tail of the return time function can be controlled in terms of the time generic points needed to achieve some uniform expanding behavior. As a consequence we obtain some rates for the decay of correlations of those maps and conditions for the validity of the Central Limit Theorem.

1. Dynamical and geometrical assumptions

Let M be a compact Riemannian manifold of dimension $d \geq 1$ with a normalized Riemannian volume $|\cdot|$, which we call *Lebesgue measure*. Let $f : M \to M$ be a C^2 local diffeomorphism for all $x \in M \setminus \mathcal{C}$, where $\mathcal{C}$ is some *critical set*, which may include points at which the derivative Df_x is degenerate, as well as points of discontinuity and points at which the derivative is infinite. We assume the following natural non-degeneracy condition on $\mathcal{C}$, which generalizes the notion of *non-flat* critical points for smooth one-dimensional maps.

Definition 1. The *critical set* $\mathcal{C} \subset M$ is *non-degenerate* if $|\mathcal{C}| = 0$ and there is a constant $\beta > 0$ such that for every $x \in M \setminus \mathcal{C}$ we have $\operatorname{dist}(x,\mathcal{C})^{\beta} \lesssim \|Df_x v\|/\|v\| \lesssim \operatorname{dist}(x,\mathcal{C})^{-\beta}$ for all $v \in T_x M$, and the functions $\log\det Df$ and $\log\|Df^{-1}\|$ are *locally Lipschitz* with Lipschitz constant $\lesssim \operatorname{dist}(x,\mathcal{C})^{-\beta}$.

We now state our two dynamical assumptions: the first is on the growth of the derivative and the second is on the approach rate of orbit to the critical set. Notice that for a linear map A, the condition $\|A\| > 1$ only provides information about the existence of *some* expanded direction, whereas the condition $\|A^{-1}\| < 1$ (i.e., $\log\|A^{-1}\|^{-1} > 0$) implies that *every* direction is expanded.

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Definition 2. We say that f is *non-uniformly expanding* if there exists $\lambda > 0$ such that

$$(*) \qquad \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df^{-1}_{f^i(x)}\|^{-1} \geq \lambda$$

for almost every $x \in M$.

Definition 3. We say that f satisfies the property of *subexponential recurrence* to the critical set if for any $\epsilon > 0$ there exists $\delta > 0$ such that for Lebesgue almost every $x \in M$

$$(**) \qquad \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leq \epsilon.$$

Here $d_\delta(x,\mathcal{C})$ denote the δ-*truncated* distance from x to $\mathcal{C}$ defined as $d_\delta(x,\mathcal{C}) = d(x,\mathcal{C})$ if $d(x,\mathcal{C}) \leq \delta$, and $d_\delta(x,\mathcal{C}) = 1$ otherwise.

For the proofs of some technical lemmas (in particular, Lemma 4) we need to fix ε satisfying certain conditions, and some of the definitions below (in particular, Definition 4) depend on this choice. We suppose therefore that some suitable ε and the corresponding δ are fixed for the rest of the paper.

Remark 1. It was proved in [2] that conditions $(*)$ and $(**)$ imply the existence of an absolutely continuous invariant probability measure μ on M. Once such a measure is given, both conditions admit very natural equivalent formulations

$$(*) \Leftrightarrow \int \log \|Df_x^{-1}\|^{-1} d\mu > 0$$

and

$$(**) \Leftrightarrow \int |\log \operatorname{dist}(x,\mathcal{C})| d\mu < \infty.$$

2. Measuring the non-uniformity

The asymptotic, non-uniform, nature of conditions $(*), (**)$ is one of the main reasons for the difficulties in studying the finer geometric structures and dynamical properties of f. To gain some control over this non-uniformity we introduce the following

Definition 4. Let $\Gamma_n = \{x : \mathcal{E}(x) > n \text{ or } \mathcal{R}(x) > n\}$, where

$$\mathcal{E}(x) = \min\left\{N \geq 1 : \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} \log \|Df^{-1}_{f^i(x)}\|^{-1} \geq \lambda/2 \quad \forall n \geq N\right\}$$

is the *expansion time* function, and

$$\mathcal{R}(x) = \min\left\{N \geq 1 : \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leq 2\varepsilon, \forall n \geq N\right\}$$

is the *recurrence time* function.

We think of $\mathcal{E}(x), \mathcal{R}(x)$ as the *waiting times* before the asymptotic behaviour kicks in. By $(*)$ and $(**)$, $\mathcal{E}$ and $\mathcal{R}$ are defined and finite a.e., and therefore $|\Gamma_n| \to 0$ as $n \to \infty$. The *rate* at which $|\Gamma_n|$ decays is, in some sense, a measure of the *non-uniformity* of f. Our main result shows that this intuition is reflected in certain geometrical properties of the dynamics of f.

3. Generalized Markov Partitions

Before stating our main theorem we introduce the geometric structures in which we are interested.

Definition 5. We say that f admits a *Markov Tower* or *Generalized Markov Partition* if there exists a ball $\Delta \subset M$, a countable partition $\mathcal{P}$ (mod 0) of Δ into topological balls U with smooth boundaries, and a return time function $R : \Delta \to \mathbb{N}$ piecewise constant on elements of $\mathcal{P}$, satisfying the following properties:

1. **Markov**: for each $U \in \mathcal{P}$ and $R = R(U)$, $f^R : U \to \Delta$ is a C^2 diffeomorphism. We let $F(x) = f^{R(x)}(x)$.
2. **Uniform expansivity**: $\exists\ \hat{\lambda} > 1$ such that $\|DF(x)^{-1}\|^{-1} \geq \hat{\lambda}$ for a.e. $x \in \Delta$. In particular, the separation time $s(x,y) = \max\{k : F^i(x), F^i(y)$ belong to the same element of $\mathcal{P}, \forall\ i \leq k\}$ is finite for a.e. pair x, y.
3. **Bounded volume distortion**: $\exists\ K > 0$ such that $\left|\frac{\det DF(x)}{\det DF(y)} - 1\right| \leq K\hat{\lambda}^{-s(F(x),F(y))}$ $\forall\ x, y$ with $s(x,y) \in [1, \infty)$.

The main difference between this and a standard Markov partition is that here the partition is not defined on the whole manifold M but only on some possibly small subset Δ, and that the Markov property is not verified after a single iterate of f but after a variable, unbounded, number of iterates which depend on the partition element. These weaker conditions make it possible to prove the existence of Generalized Markov Partition in much more general situations than those for which standard Markov partitions exist.

4. Statement of results

Theorem 1. *Let $f : M \to M$ be a transitive C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$ satisfying conditions $(*)$ and $(**)$, and suppose that there exists $\gamma > 0$ such that*

$$|\Gamma_n| = \mathcal{O}(n^{-\gamma}).$$

Then f admits a Generalized Markov Partition, and the return time function satisfies

$$|\{x : R(x) > n\}| = \mathcal{O}(n^{-\gamma}).$$

There are several possible motivations for the construction of Generalized Markov Partitions; we refer to [3, 4] for a detailed discussion and references. We mention here one implication for statistical properties of the maps, which follows from our result and from [7, 8].

Corollary 1. *Let $f : M \to M$ be a transitive C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$ satisfying conditions $(*)$ and $(**)$, and suppose that there exists $\gamma > 0$ such that $|\Gamma_n| = \mathcal{O}(n^{-\gamma})$. Then there exists an absolutely continuous, f-invariant, probability measure μ. Some finite power of f is mixing with respect to μ, and for any Hölder continuous functions φ, ψ on M we have*

$$\mathcal{C}_n = \left|\int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu\right| = \mathcal{O}(n^{-\gamma+1}).$$

Moreover, if $\gamma > 2$, then the Central Limit Theorem holds.

In particular, we obtain the following results for the two-dimensional non-uniformly expanding *Viana maps* [6, 1].

Corollary 2. *The Viana maps satisfy the Central Limit Theorem and exhibit* super-polynomial *decay of correlations, i.e.,*

$$\mathcal{C}_n = \mathcal{O}(n^{-\gamma}) \quad \forall\, \gamma > 0.$$

5. Basic strategy

We restrict ourselves here to the outline of the main steps of the proof of the theorem; the details will appear in [3] and [4]. We observe first of all that the transitivity assumption implies the existence of a point p with dense preimages, and choose some sufficiently small ball Δ_0 centred at p. This will be the domain of definition of our induced map. The idea is to consider iterates of Δ_0 until we find some n_0 such that $f^{n_0}(\Delta_0)$ completely covers Δ_0 and some bounded distortion property is satisfied. Then there is $U \subset \Delta_0$ such that $f^{n_0}(U) = \Delta_0$, and U is by definition an element of the final partition $\mathcal{P}$ with associated return time $R = n_0$. We then continue iterating the complement $\Delta_0 \setminus U$ until more good returns occur. By taking some care in the construction, this does indeed yield a Generalized Markov Partition with the required bounds on the tail of the return times. For this purpose we need some more concrete geometrical and combinatorial information regarding the time it takes for given domains to grow in size and eventually cover Δ_0, and on the geometry of the complement $\Delta_0 \setminus U$. Indeed, iterating the construction, at time n we will be dealing with the complement of an increasing number of domains corresponding to regions which have had good returns up to time n.

6. Returning to a given domain

Our first observation implies that it is sufficient for a domain to grow large enough with bounded distortion, to guarantee that a good return to Δ_0 will then occur within some fixed maximum number of iterates.

Lemma 2. $\forall\ \delta > 0$, $\exists\ N_0 \in \mathbb{N}$ *such that* $\bigcup_{j=0}^{N_0} f^{-j}(\{p\})$ *is* δ*-dense in* M *and disjoint from* $\mathcal{C}$*.*

Thus any sufficiently large ball will contain a preimage of p close to its centre, and the statement made above holds true. In paricular, it is sufficient to concentrate on the rate at which small regions grow to some fixed large scale.

7. Growing to large scale

We approach this problem through the notion of *hyperbolic times* introduced in [1]. We say that for a given $\delta_1 > 0$, k is a hyperbolic time for x if there exists a neighbourhood V_n of x, called a *hyperbolic preball*, such that $f^n(V_n)$ is a ball of radius δ_1 and the volume distortion of f^n on V_n is uniformly bounded by a given constant independent of n or x. In particular, if $\delta \ll \delta_1$, using Lemma 2, it is possible to prove

Lemma 3. $\exists\ c > 0$ *such that if* n *is a hyperbolic time for* x, $\exists$ *a neighbourhood* $U \subset V_n$ *of* x *with* $|U|/|V_n| \geq c$ *such that* $f^{n+i}(U) = \Delta_0$ *for some* $i \leq N_0$, *and* f^{n+i} *has bounded volume distortion on* U*.*

Thus, the return time is controlled locally by the occurrence of a hyperbolic time. This is naturally related to the expansion and recurrence time functions through the following

Lemma 4. *$\exists\ \theta > 0$ such that $\forall\ x$ and $n \geq \max\{\mathcal{E}(x), \mathcal{R}(x)\}$ $\exists\ \theta n$ hyperbolic times $n_1 < \cdots < n_{\theta n} < n$.*

8. Global rate of returns

Ideally we would like to be able to cover the set $\Delta_n = \Delta_0 \setminus \{R < n\}$ with disjoint hyperbolic preballs corresponding to some controlled sequence of hyperbolic times. However, we do not have enough information to carry out such a strategy and there are some technical issues as well. First of all we need to avoid points which are too close to the boundary of Δ_n. We write $\Delta_n = A_n \cup B_n$, where B_n is a small neighbourhood of $\Delta_0 \setminus \Delta_n$ in Δ_n, a kind of *buffer* zone to smooth out the complicated geometry given by the history of previous returns. In particular, the definition of B_n essentially guarantees that any hyperbolic preball $V_n(x)$ for $x \in A_n$ is completely contained in Δ_n. We let H_j denote the set of points in Δ_0 for which j is a hyperbolic time.

Lemma 5. *$\exists\ c > 0$ such that $\forall\ n \geq 1$ we have $|\bigcup_{i=0}^{N}\{x : R(x) = n + i\}| \geq c_0|A_{n-1} \cap H_n|$.*

This says that the proportion of A_{n-1} which has return time between n and $n + N$ is uniformly comparable to the proportion of points in A_n for which n is a hyperbolic time. From this we get

Lemma 6. *$\exists\ b > 0$ such that $|\Delta_{n+N}| \leq |\Delta_n| e^{-b\sum_{j=1}^{n}|A_{j-1}\cap H_j|/|A_{j-1}|}$.*

Thus the rate of decay of the $|\Delta_n|$, which is precisely the rate of decay of the tail of the return times, depends on the proportion of each A_{j-1} which has a hyperbolic time at time j. In the uniformly expanding case every iterate j is a hyperbolic time for every x, and therefore $|A_{j-1} \cap H_j|/|A_{j-1}| \equiv 1$, giving an exponential decay of the tail of the return times as expected. In our case we can only get that for all $n \geq 1$ and $A \subset M \setminus \Gamma_n$, we have $\sum_{j=1}^{n}|A \cap H_j|/|A| \geq \theta n$, as a corollary of Lemma 4.

9. Conclusions

Thus, intuitively, if the complement of Γ_n has many hyperbolic times, good returns occur exponentially fast. If $|\Gamma_n|$ decays slowly, then there will come a time that most points which have not yet returned belong to Γ_n, implying that the exponential rate of returns cannot continue until $|\Gamma_n|$ has become sufficiently small again. Thus Γ_n is a *bottleneck* slowing down the return times. This idea cannot be completely implemented in practice however, mostly because of the difference between the terms $\sum_{j=1}^{n}|A \cap H_j|/|A|$, which appear in Lemma 4, and the terms $\sum_{j=1}^{n}|A_{j-1} \cap H_j|/|A_{j-1}|$, which appear in Lemma 6. Given the abstract nature of our assumptions and the definitions of the sets A_j and B_j, the A_j may vary much more irregularly than may be expected at first sight. For example, it is possible to envisage a situation in which $\bigcap_{j=1}^{n} A_j = \emptyset$. This means that it is not possible to apply the conclusions of Lemma 4 directly to conclude that returns are occurring exponentially fast outside Γ_n. By considering various possible cases, it is nevertheless possible to obtain some good bounds in the case in which Γ_n is decaying polynomially fast, as stated in the theorem.

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