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ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS ON COMPLETE MANIFOLDS

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Abstract. In several influential works, Melrose has studied examples of non-compact manifolds M_0 whose large scale geometry is described by a Lie algebra of vector fields $\mathcal{V} \subset \Gamma(M;TM)$ on a *compactification* of M_0 to a manifold with corners M. The geometry of these manifolds—called "manifolds with a Lie structure at infinity"—was studied from an axiomatic point of view in a previous paper of ours. In this paper, we define and study an algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ of pseudodifferential operators canonically associated to a manifold M_0 with a Lie structure at infinity $\mathcal{V} \subset \Gamma(M;TM)$. We show that many of the properties of the usual algebra of pseudodifferential operators on a compact manifold extend to the algebras that we introduce. In particular, the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is a "microlocalization" of the algebra $\mathrm{Diff}^*_{\mathcal{V}}(M)$ of differential operators with smooth coefficients on M generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$. This proves a conjecture of Melrose (see his ICM 90 proceedings paper).

Introduction

In [17], Melrose has formulated a far reaching program to study the analytic properties of geometric differential operators on an open manifold M_0, provided that its large scale geometry is controlled by a Lie algebra of vector fields $\mathcal{V}$ on a compactification $M \supset M_0$. Typically, M is a manifold with corners, $M_0 = M \smallsetminus \partial M$ is obtained by removing all faces of M, and the given Lie algebra of vector fields $\mathcal{V} \subset \Gamma(M;TM)$ satisfies a number of axioms (see Section 1). This structure leads to complete metrics of bounded curvature on M_0, and ∂M is the "boundary at infinity." For example, manifolds with asymptotically Euclidean, asymptotically hyperbolic, or asymptotically complex hyperbolic ends are obtained in this way.

An important ingredient in Melrose's program mentioned above is to define a suitable pseudodifferential calculus on M_0 adapted in a certain sense to $(M,\mathcal{V})$. Melrose calls this pseudodifferential calculus a "microlocalization of $\mathrm{Diff}^*_{\mathcal{V}}(M)$," where $\mathrm{Diff}^*_{\mathcal{V}}(M)$ is the algebra of differential operators on M generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$.

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In [19] and several other papers, Melrose and his collaborators have completed this program in several important cases [6, 9, 10, 13, 14, 15, 16, 18, 19, 20, 21, 22, 32]. One of the main points is that the geometric operators on manifolds with a Lie structure at infinity identify with degenerate differential operators on the compactification M. This type of differential operators appear naturally, for example, in the study of boundary value problems on manifolds with singularities. Results in this direction were obtained also by Schulze and his collaborators, who typically worked in the framework of the Boutet de Monvel algebras. See [26, 27] and the references therein. See also [11, 24, 25].

It is desirable to present all these cases in a unified setting and to extend the results to a larger class of manifolds, namely the class of "manifolds with a Lie structure at infinity." These are open manifolds M_0 which are topologically the interior of a compact manifold M with corners, and the geometry of M_0 near ∂M is described by a Lie algebra of vector fields $\mathcal{V} \subset \Gamma(TM)$ satisfying certain axioms (see Section 1 for details). The geometrical properties of these manifolds were studied in [1]. Here we introduce an algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ of pseudodifferential operators on M_0 that is canonically associated to the manifold with a Lie structure at infinity M_0. Then we show that this algebra "microlocalizes" $\mathrm{Diff}^*_{\mathcal{V}}(M)$ in the sense that $\mathrm{Diff}^*_{\mathcal{V}}(M)$ is the algebra of all differential operators in $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ and that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ has the usual symbolic properties of algebras of pseudodifferential operators on a compact manifold. We also show that the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is invariant under the diffeomorphisms of M_0 obtained by exponentiating the vector fields $X \in \mathcal{V}$ and under conjugation with complex powers of the functions that define the faces of the compactification M of M_0.

The explicit construction of the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ microlocalizing $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ in the sense of [17] is, roughly, as follows. First, $\mathcal{V}$ defines an extension of TM_0 to a vector bundle $A \to M$ ($M_0 = M \smallsetminus \partial M$). Denote $V_r := \{d(x,y) < r\} \subset M_0^2$ and $(A)_r = \{v \in A, \|v\| < r\}$. Let $r > 0$ be less than the injectivity radius of M_0 and $V_r \ni (x,y) \mapsto (x, \tau(x,y)) \in (A)_r$ be a local inverse of the Riemannian exponential map $TM_0 \ni v \mapsto \exp_x(-v) \in M_0 \times M_0$. Let χ be a smooth function on A with support in $(A)_r$, $\chi = 1$ on $(A)_{r/2}$. For any $a \in S^m_{1,0}(A^*)$, we define

$$(1) \quad \left[a_\chi(D)u\right](x) = (2\pi)^{-n} \int_{M_0} \left(\int_{T^*_x M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x,y)) a(x,\eta) u(y)\, d\eta \right) dy.$$

The algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ is then generated linearly by the operators $a_\chi(D)$ and $b_\chi(D)\exp(X_1)\ldots\exp(X_k)$, $a \in S^{\infty}(A^*)$, $b \in S^{-\infty}(A^*)$, and $X_j \in \mathcal{V}$. (We need to introduce the operators $b_\chi(D)\exp(X_1)\ldots\exp(X_k)$, where $\exp(X_j)$ is the exponential of the vector field X_j, to make our space of operators closed under products.) The definition of the operators of order $-\infty$ is in fact the crucial part of this construction.

In [17], Melrose outlined the construction of a pseudodifferential calculus on manifolds with Lie structure at infinity provided certain manifolds with corners (blow-ups) can be constructed. (See also [31].) Our approach owes a lot to his approach when it comes to proving that $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ are algebras, but we replace the blow-ups with groupoids, using also a deep result of Crainic and Fernandes [4] on the integration of Lie algebroids. (However, to prove the original form of the conjecture from [17], the earlier results of [23] also suffice.)

This paper is an announcement. Complete proofs will be published in [2].

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1. Manifolds with Lie structure at infinity

For the convenience of the reader, let us recall the definition of Riemannian manifolds with a Lie structure at infinity and some of their basic properties [1].

In the sequel, by a *manifold* we shall always understand a C^∞-manifold *with corners*, whereas a *smooth manifold* is a C^∞-manifold *without corners*. By definition, every point p in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$ such that the transition functions are smooth up to the boundary. We then call p a point of *boundary depth at most* k and write $\operatorname{codim}(p) \leq k$. Points p with $\operatorname{codim}(p) \leq k$ but not $\operatorname{codim}(p) \leq k-1$ are said to be of *boundary depth* k. (This terminology is in agreement with the terminology for stratified spaces, if we stratify a manifold with corners by its open subfaces; see [12].) We denote by ∂M the union of all non-trivial faces of M. Usually, we write M_0 for the interior of M, i.e., $M_0 := M \smallsetminus \partial M$.

A Lie subalgebra $\mathcal{V} \subseteq \Gamma(M;TM)$ of the Lie algebra of all smooth vector fields on M is said to be *a structural Lie algebra of vector fields* provided it is a finitely generated, projective $\mathcal{C}^\infty(M)$-module and each $V \in \mathcal{V}$ is tangent to all hyperfaces of M. (We shall denote the sections of a vector bundle $V \to X$ by $\Gamma(X;V)$, unless X is understood, in which case we shall write simply $\Gamma(V)$.) By the Serre-Swan theorem [8], there exists a smooth vector bundle $A = A_{\mathcal{V}} \to M$ together with a natural map $\varrho = \varrho_{\mathcal{V}} : A \to TM$ making the following diagram commutative

$$\begin{tikzcd} A_{\mathcal{V}} \arrow[rr, "\varrho"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd} \tag{2}$$

and such that $\mathcal{V} = \varrho(\Gamma(A_{\mathcal{V}}))$.

Definition 1.1. A *Lie structure at infinity* on a smooth manifold M_0 is a triple $(M_0, M, \mathcal{V})$, where M is a compact manifold, possibly with corners, and $\mathcal{V} \subset \Gamma(M;TM)$ is a structural Lie algebra of vector fields on M with the following properties:

(a) M_0 is diffeomorphic to the interior $M \smallsetminus \partial M$ of M;
(b) If $\varrho : A \to TM$ is the anchor map defined by diagram (2), then ϱ restricts to an isomorphism $A|_{M\smallsetminus\partial M} \to TM|_{M\smallsetminus\partial M}$.

Note that for a given manifold M_0 in general there can exist many Lie structures at infinity. Examples of Lie structures at infinity were discussed in [1].

From now on, we identify M_0 with $M \smallsetminus \partial M$ and $A|_{M_0}$ with TM_0. Because A and $\mathcal{V}$ determine each other up to isomorphism, we sometimes specify a Lie structure at infinity on M by the pair (M, A). Also, it follows that $\varrho : \Gamma(M;A) \to \Gamma(M;TM)$ is injective, so we shall identify $\Gamma(M;A)$ with $\mathcal{V} = \varrho(\Gamma(M;A))$.

Elements in the enveloping algebra $\operatorname{Diff}^*_{\mathcal{V}}(M)$ of $\mathcal{V}$ are called $\mathcal{V}$*-differential operators on* M. By the injectivity of the induced structural map $\varrho_{\mathcal{V}} : \Gamma(A_{\mathcal{V}}) \to \Gamma(TM)$, the algebra of $\mathcal{V}$-differential operators can be realized as a subalgebra of all differential operators on M; in particular, they act continuously on the space $\mathcal{C}^\infty(M)$. Moreover, the order of differential operators induces a filtration $\operatorname{Diff}^m_{\mathcal{V}}(M)$, $m \in \mathbb{Z}_+$, on the algebra $\mathrm{Diff}^*_{\mathcal{V}}(M)$. Since $\mathrm{Diff}^*_{\mathcal{V}}(M)$ is a $\mathcal{C}^\infty(M)$-module, we can introduce $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to M$, $E, F \subset M \times \mathbb{C}^N$ by

$$\mathrm{Diff}^*_{\mathcal{V}}(M; E, F) := e_F M_N(\mathrm{Diff}^*_{\mathcal{V}}(M)) e_E \,, \tag{3}$$

where $e_E, e_F \in M_N(\mathcal{C}^\infty(M))$ are the projections onto E and, respectively, F. It follows that $\mathrm{Diff}^*_{\mathcal{V}}(M; E, E) =: \mathrm{Diff}^*_{\mathcal{V}}(M; E)$ is an algebra that is closed under adjoints and contains all geometric operators on M_0 that are associated to a metric on M_0 that comes from a metric on A. (See [1].)

Since any metric on A induces a natural metric on $TM_0 = A|_{M_0}$, we obtain the following definition.

Definition 1.2. A manifold M_0 with a Lie structure at infinity (M, A) and with metric g_0 on M_0 obtained by restricting a metric g from A to TM_0 is called a *Riemannian manifold with a Lie structure at infinity.*

The geometry of Riemannian manifolds (M_0, g_0) with a Lie structure at infinity has been studied in [1]. For instance, (M_0, g_0) is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor are bounded. Under additional mild assumptions, we also know that the injectivity radius $\mathrm{injrad}(p)$, viewed as a function depending on $p \in M$, is bounded below by a positive constant, or, equivalently, (M_0, g_0) is of bounded geometry in the sense of [28] and references therein. We shall denote by $r_0 := \inf_p \mathrm{injrad}(p)$ the *injectivity radius* of M_0.

On a Riemannian manifold M_0 with a uniform structure at infinity (M, A), the exponential map $\exp_p : TM_0 \to M_0$ is well defined for all $p \in M_0$ and extends to a differentiable map $\exp_p : A_p \to M$ depending smoothly on $p \in M$. A convenient way to introduce the exponential map is via the geodesic spray, as done in [1]. Similarly, any vector field $X \in \Gamma(A)$ is integrable and will map any (connected) face to itself. The resulting diffeomorphism of M_0 will be denoted ψ_X.

We shall also assume from now on that r_0, the injectivity radius of (M_0, g_0), is positive.

2. Kohn-Nirenberg quantization and pseudodifferential operators

We now introduce the algebras $\Psi_{cl,\mathcal{V}}(M_0)$ and $\Psi_{1,0,\mathcal{V}}(M_0)$ of pseudodifferential operators on M_0 adapted to the Lie structure at infinity $(M, \mathcal{V})$. We also state some of their main properties.

2.1. **Riemann-Weyl fibration.** Fix now a Riemannian metric g on the bundle A, and let $g_0 = g|_{M_0}$ be its restriction to the interior M_0 of M, defined in view of the identification $A|_{M_0} = TM_0$. We shall use this metric to trivialize all density bundles on M. Denote by $\pi : TM_0 \to M_0$ the natural projection.

Define

$$\Phi : TM_0 \longrightarrow M_0 \times M_0, \quad \Phi(v) := (x, \exp_x(-v)),\ x = \pi(v). \tag{4}$$

Recall that, for $v \in T_xM$, we have $\exp_x(v) = \gamma_v(1)$, where γ_v is the unique geodesic with $\gamma_v(0) = \pi(v) = x$ and $\gamma_v'(0) = v$. It is known that there is an open neighborhood U of the zero-section M_0 in TM_0 such that $\Phi|_U$ is a diffeomorphism onto an open neighborhood V of the diagonal $\Delta_{M_0} \subseteq M_0 \times M_0$.

To fix notation, let E be a vector bundle with a norm $\|\cdot\|$. We shall denote by $(E)_r$ the set of all vectors v of E with norm $\|v\| < r$. Our assumption that the injectivity radius r_0 of M_0 is positive gives that, for $0 < r \leq r_0$, the restriction $\Phi|_{(TM_0)_r}$ is a diffeomorphism onto an open neighborhood V_r of the diagonal Δ_{M_0}. We continue, by slight abuse of notation, to write Φ for that restriction. Following [5], we shall call Φ a *Riemann-Weyl fibration.* However, note that the Riemann-Weyl fibrations are defined in a slightly different way in [5]; the difference will be of no importance for us. The inverse of Φ is given by

$$M_0 \times M_0 \supseteq V_r \ni (x, y) \longmapsto (x, \tau(x, y)) \in (TM_0)_r\,,$$

where $\tau(x, y) \in T_xM_0$ is the tangent vector at x to the shortest geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

We shall denote by $S^m_{1,0}(E)$ the space of symbols of order m and type $(1, 0)$ on E (in Hörmander's sense) and by $S^m_{cl}(E)$ the space of classical symbols of order m on E [7, 29, 30]. These spaces are reviewed in [2] in our framework.

Let $\chi \in \mathcal{C}^\infty(A^*)$ be a smooth function that is equal to 1 on $(A^*)_r$ and is equal to 0 outside $(A^*)_{2r}$, for some $r < r_0/3$. Then, following [2], we define

$$a_\chi(D)u(x) = (2\pi)^{-n} \int_{T^*M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x, y)) a(x, \eta) u(y)\, d\eta\, dy\,.$$

This integral is an oscillatory integral with respect to the symplectic measure on T^*M_0 [7]. Alternatively, we can consider the measures on M_0 and on $T^*_xM_0$ defined by some choice of a metric on A and then we integrate first along the fibers $T^*_xM_0$ and then along M_0.

Proposition 2.1. *The map* $\sigma_{tot} : S^m_{1,0}(A^*) \to \Psi^m(M_0)/\Psi^{-\infty}(M_0)$,

$$\sigma_{tot}(a) := a_\chi(D) + \Psi^{-\infty}(M_0),$$

is independent of the choice of the function $\chi \in \mathcal{C}^\infty_c((A)_r)$ *used to define* $a_\chi(D)$.

The space of all operators of the form $a_\chi(D)$, with $a \in S^*_{1,0}(A^*)$, is not closed under composition. In order to make it closed under composition, we are going to include more operators of order $-\infty$ in our calculus.

Any vector field $X \in \Gamma(A)$ generates a global flow $\Psi_X : \mathbb{R} \times M \to M$ because X is tangent to all boundary faces of M and M is compact. Evaluating at $t = 1$ yields a diffeomorphism

$$\psi_X := \Psi_X(1, \cdot) : M \to M. \tag{5}$$

We continue to assume that the injectivity radius r_0 of our fixed manifold with a Lie structure at infinity (M, M_0, A) is strictly positive.

Definition 2.2. Fix $0 < r < r_0$ and $\chi \in \mathcal{C}^\infty_c((A)_r)$ such that $\chi = 1$ in a neighborhood of M in A. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of operators $\mathcal{C}^\infty_c(M_0) \to \mathcal{C}^\infty_c(M_0)$ generated by $a_\chi(D)$, $a \in S^m_{1,0}(A^*)$, and $b_\chi(D)\psi_{X_1} \dots \psi_{X_k}$, $b \in S^{-\infty}(A^*)$ and $X_j \in \Gamma(A)$, for all j.

Similarly, the space $\Psi^m_{cl,\mathcal{V}}(M_0)$ of *classical pseudodifferential operators generated by the Lie structure at infinity* (M, A) is obtained by using classical symbols a in the construction above.

We also obtain that $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ are algebras independent of the choices made in their definition.

Theorem 2.3. *The spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ are filtered algebras. They do not depend on the choice of the metric on A and the function χ used to define it, but depend, in general, on the Lie structure at infinity (M, A) on M_0.*

The fact that $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ are filtered algebras means that

$$\Psi^m_{1,0,\mathcal{V}}(M_0)\Psi^{m'}_{1,0,\mathcal{V}}(M_0) \subseteq \Psi^{m+m'}_{1,0,\mathcal{V}}(M_0)\,, \qquad \Psi^m_{cl,\mathcal{V}}(M_0)\Psi^{m'}_{cl,\mathcal{V}}(M_0) \subseteq \Psi^{m+m'}_{cl,\mathcal{V}}(M_0)\,,$$

for all $m, m' \in \mathbb{C} \cup \{-\infty\}$.

The proof of Theorem 2.3 is obtained by realizing $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ as the homomorphic image of the algebra $\Psi^\infty_{1,0}(\mathcal{G})$ of pseudodifferential operators on a groupoid $\mathcal{G}$ integrating the Lie algebroid A (that is, with $\Gamma(A) = \mathcal{V}$). This is possible due to the results of [4, 23, 24]. Note that this proof provides also an alternative definition of the algebras $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{cl,\mathcal{V}}(M_0)$. The advantage of our original definition, however, is that it is intrinsically formulated in terms of the geometry of M.

As for the usual algebras of pseudodifferential operators, we have the following basic property of the principal symbol.

Proposition 2.4. *The principal symbol establishes isomorphisms*

$$\sigma^{(m)} : \Psi^m_{1,0,\mathcal{V}}(M_0)/\Psi^{m-1}_{1,0,\mathcal{V}}(M_0) \to S^m_{1,0}(A^*)/S^{m-1}_{1,0}(A^*) \tag{6}$$

and

$$\sigma^{(m)} : \Psi^m_{cl,\mathcal{V}}(M_0)/\Psi^{m-1}_{cl,\mathcal{V}}(M_0) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*) \simeq \mathcal{C}^\infty(S^*A), \tag{7}$$

where S^*A is the set of unit vectors in A^*.

We have the following boundedness result.

Proposition 2.5. *Any operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ defines a continuous linear operator on $\mathcal{C}^\infty_c(M_0)$ and $\mathcal{C}^\infty(M)$. If $m = 0$, it also defines a bounded operator on $L^2(M_0)$.*

Part (i) of the following result is an analog of a standard result about the b-calculus [18], whereas the second formula is the independence of diffeomorphisms of the algebras $\Psi^\infty_{cl,\mathcal{V}}(M_0)$, in the framework of manifolds with a Lie structure at infinity. Recall that if $X \in \Gamma(A)$, we have denoted by $\psi_X := \Psi_X(1, \cdot) : M \to M$ the diffeomorphism defined by integrating X (and specializing at $t = 1$).

Proposition 2.6. *(i) Let x be a defining function of some hyperface of M. Then*

$$x^s \Psi^m_{1,0,\mathcal{V}}(M_0) x^{-s} = \Psi^m_{1,0,\mathcal{V}}(M_0) \quad \text{and} \quad x^s \Psi^m_{cl,\mathcal{V}}(M_0) x^{-s} = \Psi^m_{cl,\mathcal{V}}(M_0)$$

for any $s \in \mathbb{C}$.

(ii) Similarly,

$$\psi_X \Psi^m_{1,0,\mathcal{V}}(M_0) \psi_X^{-1} = \Psi^m_{1,0,\mathcal{V}}(M_0) \quad \text{and} \quad \psi_X \Psi^m_{cl,\mathcal{V}}(M_0) \psi_X^{-1} = \Psi^m_{cl,\mathcal{V}}(M_0),$$

for any $X \in \Gamma(A)$.

Let us notice that (ii) remains true for any diffeomorphism of M_0 that extends to an automorphism of (M, A). Recall that an autormorphism of the Lie algebroid (M, A) is a morphism of vector bundles (ϕ, ψ), $\phi : M \to M$, $\psi : A \to A$, such that ϕ and ψ are diffeomorphisms and

$$\varrho_\Gamma \circ \psi_\Gamma = \phi_* \circ \varrho_\Gamma,$$

where $\varrho_\Gamma : \Gamma(A) \to \Gamma(TM)$ and $\psi_\Gamma : \Gamma(A) \to \Gamma(A)$ are the maps defined by the anchor map ϱ and ψ, respectively, and $\phi_* : \Gamma(TM) \to \Gamma(TM)$ is given by the differential of ϕ.

A proof of the above proposition can be obtained using the corresponding results for the algebras of pseudodifferential operators on $\mathcal{G}$.

The proof of the following proposition relies on the Campbell-Hausdorff formula.

Proposition 2.7. *Let $X \in \Gamma(A)$ and denote by $a_X(\xi) = \xi(X)$ the associated linear function on A^*. Then $a_X \in S^1(A^*)$ and $a_X(D) = -\imath X$. Moreover,*

$$\{a_X(D),\ a = \text{ polynomial in each fiber }\} = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

From this we obtain the following result.

Theorem 2.8. *Let $\mathrm{Diff}(M_0)$ be the algebra of all differential operators on M_0. Then*

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0) \cap \mathrm{Diff}(M_0) = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Since $\mathrm{Diff}^*_{\mathcal{V}}(M_0) \subset \Psi^\infty_{cl,\mathcal{V}}(M_0)$, it also follows that

$$\Psi^\infty_{cl,\mathcal{V}}(M_0) \cap \mathrm{Diff}(M_0) = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Thus the algebras $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ and $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ are microlocalizations of $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$, which, together with the other properties of these algebras stated above, shows that our constructions solve a conjecture from [17].

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