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ON 3-GRADED LIE ALGEBRAS, JORDAN PAIRS AND THE CANONICAL KERNEL FUNCTION

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ABSTRACT. We present several embedding results for 3-graded Lie algebras and KKT algebras that are generated by two homogeneous elements of degrees 1 and -1. We also propose the canonical kernel function for a "universal Bergman kernel" which extends the usual Bergman kernel on a bounded symmetric domain to a group-valued function or, in terms of formal series, to an element in the formal completion of the universal enveloping algebra of the free 3-graded Lie algebra in a pair of generators.

1. INTRODUCTION

A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1.1}$$

over a field K is called a 3-*graded Lie algebra.* They include the complexification of semisimple Lie algebras of Hermitian type with gradation provided by the Harish-Chandra decomposition, Heisenberg algebras, KKT (Kantor-Koecher-Tits) algebras, among other examples (see [11], [14]).

Several commutation expressions obtained for those Lie algebras can be generalized for 3-graded Lie algebras over a field K of characteristic zero, especially those between elements of the universal enveloping algebras of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. In order to work out these relations, we introduce the free 3-graded Lie algebra $\mathfrak{g}(x,y)$ generated by elements x of degree 1 and y of degree -1. Indeed, one can manipulate formal series in the elements of $\mathfrak{g}(x,y)$ and exponentiate them to get a group, which can be handled by formal analytic methods. A major step to prove these relations is to show that the center of $\mathfrak{g}(x,y)$ is zero if $\operatorname{char} K = 0$. As a consequence, $\mathfrak{g}(x,y)$ can be realized as a subalgebra of $\mathfrak{sl}_2(K[t])$, where t is an indeterminate.

Recall that a *Kantor-Koecher-Tits algebra*, or *KKT algebra* for short [11], is a 3-graded Lie algebra satisfying

$$\text{(i) } \mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]; \text{ (ii) } \{x \in \mathfrak{g}_0 \mid [x, \mathfrak{g}_{-1}] = [x, \mathfrak{g}_1] = 0\} = 0\,. \tag{1.2}$$

They correspond to the concept of Jordan pairs written in terms of Lie algebras for $\operatorname{char} K \neq 2, 3$ (see [10], [11]). One still obtains very interesting results if condition

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(ii) above is waived. This situation can sometimes be compared to the one existing between reductive and semisimple Lie algebras.

From a 3-graded Lie algebra $\mathfrak{g}$ one obtains a KKT algebra $\mathfrak{g}^{\#}$, defining

$$\mathfrak{g}^{\#} = \mathfrak{g}' / (\mathfrak{g}'_0 \cap Z_{\mathfrak{g}'}), \tag{1.3}$$

where $\mathfrak{g}'$ is the 3-graded Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, and $Z_{\mathfrak{g}'}$ denotes its center. One has

$$\mathfrak{g}_i^{\#} \cong \mathfrak{g}_i,\ i = -1, 1, \quad \mathfrak{g}_0^{\#} \cong \mathfrak{g}'_0 / (\mathfrak{g}'_0 \cap Z_{\mathfrak{g}'}) \cong \operatorname{ad}_{\mathfrak{g}'} \mathfrak{g}'_0, \quad \mathfrak{g}'_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]. \tag{1.4}$$

According to Corollary 2.2 and Theorem 2.3, $\mathfrak{g}(x, y)$ is isomorphic to $\mathfrak{g}^{\#}(x, y)$, the free KKT algebra in a pair of variables, if $\operatorname{char} K = 0$. From the theory of Jordan pairs, it is known that, for $\operatorname{char} K \neq 2, 3$, $\mathfrak{g}^{\#}(x, y)$ can be embedded into $\mathfrak{sl}_2(K[t])$ (cf Theorem 2.1, item (a)); in contrast, this is always false for $\mathfrak{g}(x, y)$ if K has nonzero characteristic greater than 3 (cf. Remark 2.4).

In at least two important instances of the theory, the canonical kernel function appears as a fundamental concept. In the setting of Lie groups of Hermitian type, it can be applied to the description of reproducing kernels for Hilbert spaces of holomorphic functions associated with the holomorphic discrete series representations (see [2]). In terms of free 3-graded Lie algebras, it provides the zero degree part in the formal Harish-Chandra decomposition of the product of certain exponentials. As a result, one obtains commutation relations for the universal enveloping algebra of a 3-graded Lie algebra, which generalize well-known expressions for $\mathfrak{sl}_2(K)$ (see [3]).

2. The free case

We say that a Lie algebra $\mathfrak{g}(x, y)$ over K is the *free 3-graded Lie algebra generated by variables x of degree 1 and y of degree -1 (or freely generated by the pair (x, y))* if, for any 3-graded Lie algebra $\mathfrak{h}$ over K and elements $\overline{x} \in \mathfrak{h}_1$ and $\overline{y} \in \mathfrak{h}_{-1}$, there is a unique homomorphism of graded Lie algebras $\psi : \mathfrak{g}(x, y) \to \mathfrak{h}$ such that $\psi(x) = \overline{x}$ and $\psi(y) = \overline{y}$. Clearly $\mathfrak{g}(x, y)$ exists and it is unique up to an isomorphism of graded Lie algebras.

Recall that $\mathfrak{sl}_2(K[t])$ is also a 3-graded Lie algebra over K with respect to the gradation

$$\begin{aligned}
\mathfrak{sl}_2(K[t])_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ p(t) & 0 \end{pmatrix} \ \middle|\ p(t) \in K[t] \right\}, \\
\mathfrak{sl}_2(K[t])_0 &= \left\{ \begin{pmatrix} p(t) & 0 \\ 0 & -p(t) \end{pmatrix} \ \middle|\ p(t) \in K[t] \right\}, \\
\mathfrak{sl}_2(K[t])_1 &= \left\{ \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} \ \middle|\ p(t) \in K[t] \right\}, \\
\mathfrak{sl}_2(K[t])_i &= \{0\},\ i \in \mathbb{Z} - \{-1, 0, 1\}.
\end{aligned}$$

We have the following relations between $\mathfrak{g}(x, y)$ and $\mathfrak{g}^{\#}(x, y)$, the free KKT algebra in a pair of generators.

Theorem 2.1 ([3]). *Suppose* $\operatorname{char} K \neq 2,3$. *Let* $\pi : \mathfrak{g}(x,y) \to \mathfrak{g}^{\#}(x,y)$ *be the natural projection, and let* $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{sl}_2(tK[t])$ *be the homomorphism of graded Lie algebras defined by*

$$\mathfrak{i}(x) = tE = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{i}(y) = tF = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

(a) *There exists a monomorphism*

$$\mathfrak{i}^{\#} : \mathfrak{g}^{\#}(x,y) \to \mathfrak{sl}_2(tK[t])$$

of graded Lie algebras such that

$$\mathfrak{i} = \mathfrak{i}^{\#} \circ \pi.$$

(b) $Z_{\mathfrak{g}(x,y)} = \ker \mathfrak{i} = \ker \pi = \text{linear span}\{\, [\,[x,y], (\operatorname{ad} x \operatorname{ad} y)^i\,[x,y]\,] \mid i \geqslant 1\}$.

Corollary 2.2 ([3]). *If* $\operatorname{char} K \neq 2,3$, *the following statements are equivalent:*

(a) $\mathfrak{g}(x,y) = \mathfrak{g}^{\#}(x,y)$ ($\mathfrak{g}(x,y)$ *is a KKT algebra*).
(b) *The center of* $\mathfrak{g}(x,y)$ *is zero.*
(c) *The homomorphism of graded Lie algebras* $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{sl}_2(tK[t])$ *defined by* $\mathfrak{i}(x) = t\,E$, $\mathfrak{i}(y) = t\,F$ *is a monomorphism.*

When the field has characteristic zero, the above statements are satisfied, as shown next:

Theorem 2.3 ([3]). *If* $\operatorname{char} K = 0$, *the center of* $\mathfrak{g}(x,y)$ *is zero.*

Proof. Let $I : \mathfrak{g}(x,y) \to \mathfrak{g}(x,y)$ be the identity map. We define

$$A_i : \mathfrak{g}(x,y) \to \mathfrak{g}(x,y),\ i \in \mathbb{N},$$

as $A_0 = I$ on $\mathfrak{g}(x,y)_1$, $-I$ on $\mathfrak{g}(x,y)_{-1}$ and zero on $\mathfrak{g}(x,y)_0$,

$$A_i = \operatorname{ad}((\operatorname{ad} x \operatorname{ad} y)^{i-1}[x,y]),\ i \geqslant 1,$$

and write $A = A_1$ for short. It is enough to show that

$$[\,[x,y],\ (\operatorname{ad} x \operatorname{ad} y)^{m-1}[x,y]\,] = 0$$

for any positive integer m, in view of Theorem (2.1), item (b).

We prove it by induction. For $m = 1$ it is trivial. Suppose it is valid for $m \leqslant p$. Then the following identities hold:

(i) $A_{i+1}x = AA_ix$, $p \geqslant i \geqslant 0$.
(ii) $A_{i+1}y = -AA_iy$, $i \geqslant 0$.
(iii) If $i + j = p$ for integers $i,j \geqslant 0$, then

$$[A_{i+1}x, A_jy] = A[x, A_py] + [A_ix, A_{j+1}y].$$

We prove (iii). This is obviously valid if $p = 1$. Now suppose $p \geqslant 2$. One has

$$[A_{i+1}x, A_jy] = [AA_ix, A_jy] = A[A_ix, A_jy] + [A_ix, A_{j+1}y].$$

But

$$\begin{aligned} A[A_ix, A_jy] &= A[A_iA_jy, x] + AA_i[x, A_jy] = A[A_iA_jy, x] \\ &= (-1)^{j-1}A[A_iA^jy, x] = (-1)^{j-1}A[A^jA_iy, x] \\ &= (-1)^{i+j}A[A^jA^iy, x] = -A[A_py, x] = A[x, A_py]. \end{aligned}$$

Hence

$$[A_{i+1}x, A_jy] = A[x, A_py] + [A_ix, A_{j+1}y].$$

Back to the proof of the theorem, one has for $m = p+1$:

$$\begin{aligned} A_{p+1}[x,y] &= [A_{p+1}x, y] + [x, A_{p+1}y] = -[A_{p+1}x, A_0 y] + [x, A_{p+1}y] \\ &= -(p+1)A[x, A_p y] - [A_0 x, A_{p+1}y] + [x, A_{p+1}y] \ \ \text{(by (iii))}. \end{aligned}$$

Therefore

$$(p+2)A_{p+1}[x,y] = -[x, A_{p+1}y] + [x, A_{p+1}y] = 0$$

and the theorem is proved. □

Remark 2.4. The following construction of $\mathfrak{g}(x,y)$ over K, $\operatorname{char} K = p > 3$, which also works in characteristic zero, has been pointed out to us by E. Zelmanov: Let

$$\mathfrak{sl}_2(K[t]) \oplus K[t]$$

be the 3-graded Lie algebra such that $\mathfrak{sl}_2(K[t])$ is graded as before, the elements of $K[t]$ are central and homogeneous of degree 0 and

$$[t^i a, t^j b] = t^{i+j}[a,b] + i\,\delta_{i+j,\,p}\,(a,b)\,t^{i+j},$$

where $a, b \in \mathfrak{sl}_2(K)$, $(\ ,\ )$ denotes the Killing form of $\mathfrak{sl}_2(K)$, and $\delta_{k,\,p} = 1$ if $k = 0 \,\mathbf{mod}\, p$, $\delta_{k,\,p} = 0$ otherwise. (Notice its resemblance to an affine Lie algebra.) In view of the previous development, it is clear that the 3-graded subalgebra spanned by

$$x = t\,E, \quad y = t\,F,$$

gives a realization of $\mathfrak{g}(x,y)$. In particular, for $\operatorname{char} K > 3$, $\mathfrak{g}(x,y)$ has a nontrivial center and does not satisfy Corollary 2.2.

In what follows, we restrict our attention to the theory in characteristic zero.

We notice that $\mathfrak{g}(x,y)$ and its universal enveloping algebra $\mathcal{U}(\mathfrak{g}(x,y))$ admit an $\mathbb{N}$-gradation given by the sum of occurrences of x and y in each monomial, which is called *total gradation* in either case. We consider here their formal completions $\widehat{\mathfrak{g}}(x,y)$ and $\widehat{\mathcal{U}}(\mathfrak{g}(x,y))$, which consist of formal series with finitely many terms for each degree. Similarly the *total gradation* or $\mathbb{N}$-gradation of $\mathfrak{sl}_2(K[t])$ is the one whose homogeneous elements of degree n are the matrices over scalar multiples of t^n having trace zero, and so forth. We have the following useful characterization of $\mathfrak{g}(x,y)$ and its completion $\widehat{\mathfrak{g}}(x,y)$:

Corollary 2.5 ([3])**.** *Let $\mathfrak{g}(x,y)$ be the free 3-graded Lie algebra over K, $\operatorname{char} K = 0$, generated by variables x of degree 1 and y of degree -1, endowed with the total gradation defined by the sum of occurrences of x and y in each Lie monomial.*

(a) *The homomorphism $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{sl}_2(t\,K[t])$ given by*

$$\mathfrak{i}(x) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{i}(y) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

is a monomorphism of $\mathbb{N}$-graded Lie algebras.

(b) *This monomorphism extends to a monomorphism*

$$\widehat{\mathfrak{i}} : \widehat{\mathfrak{g}}(x,y) \to \mathfrak{sl}_2(t\,K[[t]])$$

of Lie algebras, where $\widehat{\mathfrak{g}}(x,y)$ denotes the completion of $\mathfrak{g}(x,y)$ with respect to the total gradation.

(c) *Let* $\mathfrak{G}(x,y)$ *and* $SL_2^*(K[[t]])$ *be the groups obtained formally exponentiating* $\widehat{\mathfrak{g}}(x,y)$ *and* $\mathfrak{sl}_2(tK[[t]])$ *inside* $\widehat{\mathfrak{U}}(\mathfrak{g}(x,y))$ *and* $M_2(K[[t]])$, *respectively. Then the diagram*

$$\begin{tikzcd} \widehat{\mathfrak{g}}(x,y) \arrow[r, "\widehat{\mathfrak{i}}"] \arrow[d, "\exp"'] & \mathfrak{sl}_2(tK[[t]]) \arrow[d, "\exp"] \\ \mathfrak{G}(x,y) \arrow[r, "\mathfrak{I}"'] & SL_2^*(K[[t]]) \end{tikzcd}$$

is commutative, the vertical maps are bijective, and the map $\mathfrak{I}$ *is a monomorphism of groups.*

3. The canonical kernel function

The canonical kernel function is a generalization of the usual Bergman kernel function of a bounded symmetric domain $\mathcal{D}$ but it is group-valued instead. It has been introduced by I. Satake in [12], [13], [14] through several distinct contexts. It is related to different topics such as the geometry of $\mathcal{D}$, the contravariant form on highest weight modules, the Harish-Chandra decomposition for groups of Hermitian type and also reproducing kernels for families of holomorphic functions in the "holomorphic discrete series" (which includes the Hilbert space of square-integrable holomorphic functions on $\mathcal{D}$ as a special case).

Let $\mathfrak{g}$ be a real semisimple Lie algebra of Hermitian type. By Hermitian type we mean that for any Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ there is an element H_0 in the center $\mathfrak{c}$ of $\mathfrak{k}$ such that $\operatorname{ad} H_0$ is a complex structure on $\mathfrak{p}$. Fix one of them and let θ denote the corresponding Cartan involution. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. Let $\mathfrak{g}_c$, $\mathfrak{k}_c$, $\mathfrak{c}_c$, $\mathfrak{h}_c$, $\mathfrak{p}_c$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{c}$, $\mathfrak{h}$, $\mathfrak{p}$, and $\mathfrak{p}_+$, $\mathfrak{p}_-$ the eigenspaces of $\operatorname{ad}_{\mathfrak{p}_c} H_0$ with eigenvalues i and $-i$, respectively. Let G_c be the connected simply connected Lie group with Lie algebra $\mathfrak{g}_c$ and P_+, P_-, $\mathcal{K}_c$, G, $\mathcal{K}$ the analytic subgroups of G_c with Lie algebras $\mathfrak{p}_+$, $\mathfrak{p}_-$, $\mathfrak{k}_c$, $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The exponential map of G_c induces a holomorphic diffeomorphism of $\mathfrak{p}_+$ ($\mathfrak{p}_-$) onto P_+ (P_-) (see [8]).

By a theorem of Harish-Chandra, the map

$$\psi : P_+ \times \mathcal{K}_c \times P_- \to G_c\,, \qquad \psi(p,k,q) = pkq$$

is a holomorphic diffeomorphism onto an open dense subset of G_c and $G \subset P_+\mathcal{K}_c P_-$. In general, given $g \in \operatorname{Im}\psi$, we write $g = g_+\, g_0\, g_-$ for the "$P_+\mathcal{K}_c P_-$" decomposition of g, where $g_+ \in P_+$, $g_0 \in \mathcal{K}_c$ and $g_- \in P_-$.

We have the following open holomorphic embeddings:

$$\mathcal{D} = G/\mathcal{K} \simeq G\mathcal{K}_c P_-/\mathcal{K}_c P_- \hookrightarrow P_+\mathcal{K}_c P_-/\mathcal{K}_c P_- \simeq P_+ \simeq \mathfrak{p}_+\,. \tag{3.1}$$

One can prove that the image of the composition of these embeddings is bounded with respect to the metric on $\mathfrak{p}_+$ induced by the Killing form of $\mathfrak{g}_c$ and a holomorphically symmetric connected open subset of $\mathfrak{p}_+$, which is known as the Harish-Chandra realization of $\mathcal{D}$ as a bounded symmetric domain (see [14]).

The *canonical kernel function* of a bounded symmetric domain in its Harish-Chandra realization ([14]) is given by

$$\kappa : \mathcal{D} \times \mathcal{D} \to \mathcal{K}_c, \quad \kappa(z,w) = ((\exp -\overline{w}\, \exp z)_0)^{-1} \in \mathcal{K}_c, \tag{3.2}$$

which is well defined on $\mathcal{D} \times \mathcal{D}$ because, for $z, w \in \mathcal{D}$,

$$\exp -\overline{w} \exp z \in (\overline{G\mathcal{K}_c P_-})^{-1} G\mathcal{K}_c P_- = P_+\mathcal{K}_c G\mathcal{K}_c P_- = P_+\mathcal{K}_c P_-.$$

In other words, $\kappa(z, w)$ is the inverse of the $\mathcal{K}_c$-part of

$$\exp -\overline{w} \exp z \tag{3.3}$$

in the "$P_+\mathcal{K}_c P_-$" (also called Harish-Chandra) decomposition.

It is straightforward to verify that:

(i) $\kappa(0, 0) = e$,
(ii) $\kappa(z, w)$ is holomorphic in z,
(iii) $\kappa(z, w) = \overline{\kappa(w, z)^{-1}}$,
(iv) $\kappa(gz, gw) = J(g, z)\kappa(z, w)\overline{J(g, w)^{-1}}$, $g \in G$, $z, w \in \mathcal{D}$, where J is called the *canonical automorphy factor* of G, given by

$$J : G \times \mathcal{D} \to \mathcal{K}_c, \ J(g, z) = (g \exp z)_0.$$

We recall that the canonical kernel function can, alternatively, be defined by properties (i)–(iv).

Theorem 3.1 ([2]). *Let $\mathcal{D}$ be a bounded symmetric domain in its Harish-Chandra realization, with notation as above. Then*

$$\kappa(z, w) = \exp \frac{\log(1 - \operatorname{ad} z \operatorname{ad} \overline{w}/2)}{\operatorname{ad} z \operatorname{ad} \overline{w}/2}[z, \overline{w}], \qquad z, w \in \mathcal{D}.$$

The series under the exponential sign converges uniformly inside compact subsets of $\mathcal{D} \times \mathcal{D}$.

The series above should be understood as the formal development of $(\log(1 - x))/x$, where x is replaced by the linear operator $\operatorname{ad} z \operatorname{ad} \overline{w}/2$ and the output is evaluated at $[z, \overline{w}]$. We shall make use of this notation again (see Corollary 4.1).

The relation between the canonical kernel function and the Bergman kernel of $\mathcal{D}$ is the following ([14]):

Proposition 3.2. *The Bergman kernel $K(z, w)$ of $\mathcal{D}$ is given by*

$$K(z, w) = \frac{1}{\operatorname{vol} \mathcal{D}} \det \operatorname{Ad}_{\mathfrak{p}_+}^{-1} \kappa(z, w).$$

The volume $\operatorname{vol} \mathcal{D}$ above is the one relative to the Euclidean metric on $\mathfrak{p}_+$ induced by the Killing form of $\mathfrak{g}_c$. For the sake of completeness, we notice that ([2], [14]):

$$\begin{aligned}\operatorname{Ad}_{\mathfrak{p}_+} \kappa(z, w) &= 1 - \operatorname{ad} z \operatorname{ad} \overline{w} + \frac{1}{4}(\operatorname{ad} z)^2 (\operatorname{ad} \overline{w})^2 \\ &= 1 - \operatorname{ad}[z, \overline{w}] + \frac{1}{2}(\operatorname{ad}[z, \overline{w}])^2 - \frac{1}{4}\operatorname{ad}[z, [\overline{w}, [z, \overline{w}]]] \quad \text{on } \mathfrak{p}_+.\end{aligned}$$

We have considered here the representation $\pi = \det^{-1} \operatorname{Ad}_{\mathfrak{p}_+}$ of $\mathcal{K}$. In their celebrated paper ([7]), A. Korányi and J. Faraut have studied reproducing kernels for holomorphic discrete series representations as they are induced by characters of $\mathcal{K}$ (the scalar case). As A. Korányi remarked, " *The analytic problems ... are meaningful and interesting for the vector-valued case too, but there is very little known about them at present ...* " (see [6], page 259). In ([1]) and later, independently in ([2]), the reproducing kernel of a general vector-valued holomorphic discrete series representation is shown to be equal to the representation calculated at the canonical kernel function, up to a multiplicative factor. This description also appears, in implicit form, earlier in [9], [15]. The multiplicative factor has been calculated explicitly in [2] in terms of the formal degree of the representation in the holomorphic discrete series, the degree of its irreducible $\mathcal{K}$-type having the same highest weight and the Euclidean geometry of the domain.

The above results motivate us to regard the canonical kernel function as a natural candidate for a "universal Bergman kernel" in the setting of Lie groups of Hermitian type. Next we go over this concept in the context of free 3-graded Lie algebras.

4. The formal canonical kernel function

Here K denotes a field of characteristic 0. From Corollary 2.5(c) it follows

Corollary 4.1 ([3]). *In $\mathfrak{G}(x,y)$ one has*

$$\exp -x \exp y = \exp\big((1 - \operatorname{ad} y \operatorname{ad} x/2)^{-1} y\big)\, \kappa(x,y)\, \exp\big(-(1 - \operatorname{ad} x \operatorname{ad} y/2)^{-1} x\big),$$

where

$$\text{(i)} \qquad \kappa(x,y) = \exp\Big(\frac{\log(1 - \operatorname{ad} x \operatorname{ad} y/2)}{\operatorname{ad} x \operatorname{ad} y/2}[x,y]\Big),$$

which can be expanded as

$$\text{(ii)} \qquad \kappa(x,y) = \sum_{m\geqslant 0} \frac{(-1)^m}{m!} \prod_{i=1}^{m} \big([x,y] - (m-i)\operatorname{ad} x \operatorname{ad} y/2\big).$$

In (ii), *for $u, v \in \mathcal{U}(\mathfrak{g}(x,y))$ and $U, V \in End_K\big(\mathcal{U}(\mathfrak{g}(x,y))\big)$, the product in*

$$\mathcal{U}(\mathfrak{g}(x,y)) \oplus End_K\big(\mathcal{U}(\mathfrak{g}(x,y))\big)$$

is defined by

$$(u \oplus U)(v \oplus V) = (uv + U(v)) \oplus (uV + U \circ V), \quad \textit{and}$$

$$\prod_{i=1}^{n} a_i = (\cdots((a_1 a_2) a_3) \cdots a_n), \quad a_i \in \mathcal{U}(\mathfrak{g}(x,y)) \oplus End_K\big(\mathcal{U}(\mathfrak{g}(x,y))\big).$$

The product above is interpreted as 1 for $n < 1$. Now we fix a general 3-graded Lie algebra $\mathfrak{g}$ over K. From Corollary 4.1, one obtains commutation relations for powers of elements of $\mathfrak{g}_{-1}, \mathfrak{g}_1 \subset \mathcal{U}(\mathfrak{g})$. Let $z \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$. Corollary 4.1 implies that ([3])

$$\text{(4.1)} \qquad \frac{z^i}{i!}\frac{w^j}{j!} = \sum_{m=0}^{\min(i,j)} \sum_{\substack{m_1+m_2+m_3=m \\ m_1,\, m_2,\, m_3 \geqslant 0}} A_{m_1}^{j-m} B_{m_2} C_{m_3}^{i-m}, \quad \text{where}$$

$$A_0^0 = 1\,, \quad A_m^0 = 0, \quad m > 0\,.$$
$$A_m^k = \frac{1}{k!} \sum_{\substack{n_1+\cdots+n_k=m\\ n_1,\ldots,n_k \geqslant 0}} ((-\operatorname{ad} w \operatorname{ad} z/2)^{n_1} w) \cdots ((-\operatorname{ad} w \operatorname{ad} z/2)^{n_k} w),\ k > 0\,, m \geqslant 0.$$

$$B_m = \frac{1}{m!} \prod_{i=1}^{m} ([z,w] - (m-i) \operatorname{ad} z \operatorname{ad} w/2)\,, \quad m \geqslant 0.$$

$$C_0^0 = 1\,, \quad C_m^0 = 0, \quad m > 0\,.$$
$$C_m^k = \frac{1}{k!} \sum_{\substack{n_1+\cdots+n_k=m\\ n_1,\ldots,n_k \geqslant 0}} ((-\operatorname{ad} z \operatorname{ad} w/2)^{n_1} z) \cdots ((-\operatorname{ad} z \operatorname{ad} w/2)^{n_k} z), \quad k > 0\,, m \geqslant 0.$$

For instance, for $\mathfrak{g} = \mathfrak{sl}_2(K)$ and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z = E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad w = F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

one obtains the classical commutation relations:

$$\frac{E^i}{i!}\frac{F^j}{j!} = \sum_{m=0}^{\min(i,j)} \frac{F^{j-m}}{(j-m)!} \binom{H-i-j+2m}{m} \frac{E^{i-m}}{(i-m)!} \quad .$$

5. 3-graded Lie algebras in a pair of generators

We assume throughout this section that the field K has characteristic 0. Given $p(t) \in K[t]$, we write $\langle p(t)\rangle \equiv p(t)\cdot K[t]$.

Theorem 5.1 ([5]). *Any 3-graded Lie algebra generated by an element of degree 1 and another of degree -1 over a field K of characteristic zero can be realized as a 3-graded Lie subalgebra of $\mathfrak{sl}_2(K[t]\,/\langle p(t)\rangle)$ for some $p(t) \in K[t]$. Moreover, it is symmetric if and only if it is isomorphic, as a graded Lie algebra, to the 3-graded Lie subalgebra generated by*

$$(t + \langle p(t)\rangle)E \text{ and } (t + \langle p(t)\rangle)F$$

inside $\mathfrak{sl}_2(K[t]\,/\langle p(t)\rangle)$ for some $p(t) \in K[t]$.

The characterization obtained above can be used to find a classification of 3-graded Lie algebras in a pair of generators over $\mathbb{C}$ ([4]). Furthermore, since the same 3-graded Lie algebra can be embedded in $\mathfrak{sl}_2(K[t]\,/\langle p(t)\rangle)$ in different ways, it can be of interest to understand how it depends on the chosen polynomial $p(t) \in K[t]$ and representatives for the generators in $\mathfrak{sl}_2(K[t]\,/\langle p(t)\rangle)$. The following example illustrates that issue.

Let $p(t) = t^3 - ct,\ c \in \mathbb{C}$,

$$x = (t + \langle p(t)\rangle)\,E\,; \qquad y = (t + \langle p(t)\rangle)\,F,$$

and let $\mathfrak{g}$ be the 3-graded Lie algebra spanned by x and y in $\mathfrak{sl}_2(K[t]\,/\langle p(t)\rangle)$. Then

$$[x,y] \equiv h,\ [h,x] = 2\,c\,x;\ [h,y] = -2\,c\,y,$$

which shows that $\mathfrak{g}$ is a linear combination of x, y, h.

If $c \neq 0$, then it is easy to see that x, y, h can be normalized in order to produce

$$[x, y] = h,\ [h, x] = 2x;\ [h, y] = -2y.$$

In other words, $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. On the other hand, if $c = 0$, we have

$$[x, y] = h,\ [h, x] = [h, y] = 0,$$

and $\mathfrak{g}$ is the 3-dimensional Heisenberg algebra.

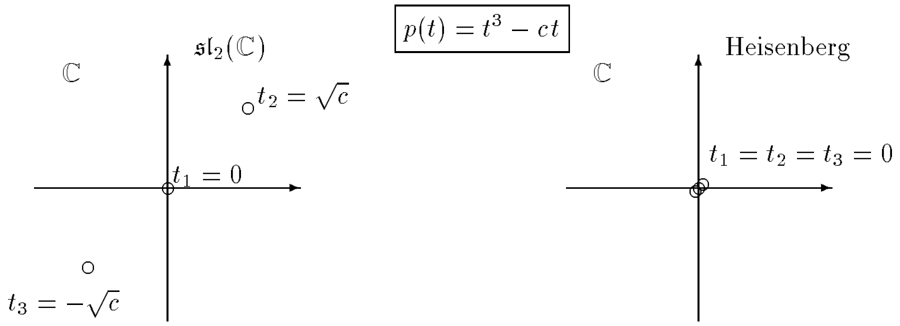

FIGURE 1. Dependence of $\mathfrak{g}$ on the roots of $p(t)$.

One can realize from the example above that the polynomial dependence of the Lie algebra occurs at the level of root multiplicities rather than with respect to the polynomial itself. Indeed, such behavior occurs in the general case. Moreover, the polynomial used in the symmetric case can be conveniently chosen, as described next:

Theorem 5.2 ([4]). (a) *Let $\mathfrak{g}$ be a symmetric 3-graded Lie algebra in a pair of generators over $\mathbb{C}$. Then $\mathfrak{g}$ is isomorphic to the 3-graded Lie subalgebra of $\mathfrak{sl}_2(\mathbb{C}[t]/\langle p(t)\rangle)$ generated by*

$$(t + \langle p(t)\rangle)E \text{ and } (t + \langle p(t)\rangle)F$$

for some $p(t) = t^m q(u)$, $u = t^2$, $q(u) \in K[u]$, $q(0) \neq 0$.

If $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$ is a Levi decomposition of $\mathfrak{g}$ as the direct sum of a semisimple subalgebra $\mathfrak{l}$ and the radical $\mathfrak{n}$ of $\mathfrak{g}$, then $\mathfrak{n}$ is nilpotent and

$$\mathfrak{l} \cong \bigoplus_{i=1}^{k} \mathfrak{sl}_2(\mathbb{C}),$$

where k is the number of distinct roots of $q(u)$.

(b) *Let $\mathfrak{g}_i$, $i = 1, 2$, be generated by*

$$(t + \langle p_i(t)\rangle)E \text{ and } (t + \langle p_i(t)\rangle)F$$

in $\mathfrak{sl}_2(\mathbb{C}[t]/\langle p_i(t)\rangle)$ for polynomials $p_i(t)$ such that

$$p_1(t) = t^l (u - a_1)^{m_1} \cdots (u - a_k)^{m_k},\ a_i \neq a_j \neq 0 \text{ for } i \neq j,\ u = t^2,$$

$$p_2(t) = t^{\tilde{l}} (u - b_1)^{n_1} \cdots (u - b_{\tilde{k}})^{n_{\tilde{k}}},\ b_i \neq b_j \neq 0 \text{ for } i \neq j,\ u = t^2.$$

Then $\mathfrak{g}_i$ are isomorphic as Lie algebras iff $k = \tilde{k}$, $l = \tilde{l}$ and the sequences (m_j) and (n_j) coincide after some reordering. In that case, the isomorphism can be chosen to preserve the gradation.

Such results raise some questions: suppose X is an algebraic variety over $\mathbb{C}$, A is a subalgebra of the regular functions on X and I is an ideal of A. Consider a finitely generated 3-graded Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}_2(A/I)$ over $\mathbb{C}$. How does $\mathfrak{g}$ depend on the zero set of I and the geometry of X (possibly under extra hypotheses)?

There seem to be interesting mathematical connections among these concepts far beyond the results of Theorem 5.2.

References

[1] M. G. Davidson and R. C. Fabec, *Geometric realizations for highest weight representations*, Contemp. Math., vol. 191, 1995, pp. 13–31. MR **97a:**22018

[2] M. P. De Oliveira, *Some formulas for the canonical kernel function*, Geom. Dedicata **86** (2001), no. 1, 227–247. MR **2002i:**22017

[3] M. P. De Oliveira, *On commutation relations for* 3*-graded Lie algebras*, New York J. Math. **7** (2001), 71–86. MR **2002k:**17056

[4] M. P. De Oliveira, *On* 3*-graded Lie algebras in a pair of generators: a classification*, J. Pure Appl. Algebra **178** (2003), no. 1, 73–85.

[5] M. P. De Oliveira and O. O. Luciano, *A characterization of* 3*-graded Lie algebras generated by a pair*, J. Pure Appl. Algebra **176** (2002), no. 2-3, 175–194. MR **2003i:**17028

[6] J. Faraut et al, *Analysis and Geometry on Complex Homogeneous Domains*, Progress in Mathematics, vol. 185, Birkhäuser, 2000. MR **2001f:**32036

[7] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89. MR **90m:**32049

[8] Harish-Chandra, *Representations of semisimple Lie groups VI. Integrable and Square-Integrable Representations*, Amer. J. Math. **78** (1956), 564–628. MR **18:**490d

[9] R. P. Langlands, *The dimension of spaces of automorphic forms*, Amer. J. Math. **85** (1963), 99–125. MR **27:**6286

[10] O. Loos, *Jordan Pairs*, Lect. Notes Math, vol. 460, Springer, Berlin-Heidelberg-New York, 1975. MR **56:**3071

[11] E. Neher, *Generators and Relations for* 3*-Graded Lie Algebras*, J. Algebra **155** (1993), 1–35. MR **94b:**17012

[12] I. Satake, *On unitary representations of a certain group extension*, Sugaku, Math. Soc., Japan **21** (1969), 241–253. MR **43:**2164

[13] ———, *Factors of automorphy and Fock representations*, Adv. in Math. **7** (1971), no. 2, 83–110. MR **50:**541

[14] ———, *Algebraic structures of symmetric domains*, Iwanami Shoten, Tokyo and Princeton University Press, Princeton, NJ, 1980. MR **82i:**32003

[15] N. Wallach, *The analytic continuation of the discrete series I*, Trans. Amer. Math. Soc. **251** (1979), 1–17. MR **81a:**22009

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