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AUTOMORPHIC FORMS ON PGSp(2)

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ABSTRACT. The theory of lifting of automorphic and admissible representations is developed in a new case of great classical interest: Siegel automorphic forms. The self-contragredient representations of PGL(4) are determined as lifts of representations of either symplectic PGSp(2) or orthogonal SO(4) rank two split groups. Our approach to the lifting uses the global tool of the trace formula together with local results such as the fundamental lemma. The lifting is stated in terms of character relations. This permits us to introduce a definition of packets and quasi-packets of representations of the projective symplectic group of similitudes PGSp(2), and analyse the structure of all packets. All representations, not only generic or tempered ones, are studied. Globally we obtain a multiplicity one theorem for the discrete spectrum of the projective symplectic group PGSp(2), a rigidity theorem for packets and quasi-packets, determine all counterexamples to the naive Ramanujan conjecture, and compute the multiplicity of each member in a packet or quasi-packet in the discrete spectrum. The lifting from SO(4) to PGL(4) amounts to establishing a product of two representations of GL(2) with central characters whose product is 1. The rigidity theorem for SO(4) amounts to a strong rigidity statement for a pair of representations of $\mathrm{GL}(2,\mathbb{A})$.

According to the "principle of functoriality", "Galois" representations $\rho : L_F \to {}^LG$ of the hypothetical Langlands group L_F of a global field F into the complex dual group ${}^LG$ of a reductive group $\mathbf{G}$ over F should parametrize "packets" of automorphic representations of the adèle group $\mathbf{G}(\mathbb{A})$. Thus a homomorphism $\lambda : {}^LH \to {}^LG$ of complex dual groups should give rise to lifting of automorphic representations π_H of $\mathbf{H}(\mathbb{A})$ to those π of $\mathbf{G}(\mathbb{A})$.

We report here on the work of [F1]. It proves the existence of the expected lifting of automorphic representations of the projective symplectic group of similitudes $\mathbf{H} = \mathrm{PGSp}(2)$ to those on $\mathbf{G} = \mathrm{PGL}(4)$. The image is the set of the self-contragredient representations of PGL(4) which are not lifts of representations of the rank two split orthogonal group SO(4).

The global lifting is defined by means of local lifting. We define the local lifting in terms of character relations. This permits us to introduce a definition of packets and quasi-packets of representations of PGSp(2) as the sets of representations that

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occur in these relations. Our main local result is that packets exist and partition the set of tempered representations. We give a detailed description of the structure of packets (in Sect. 4).

Our global results include a detailed description of the structure of the global packets and quasi-packets (the latter are almost everywhere nontempered). We obtain a *multiplicity one theorem for the discrete spectrum of* PGSp(2), a *rigidity theorem for packets and quasi-packets*, determine all *counterexamples to the naive Ramanujan conjecture*, and compute the *multiplicity of each member in a packet or quasi-packet in the discrete spectrum*, in Sect. 5.

Interesting phenomena of instability are described in Sect. 6. Sect. 7 discusses the special case of generic representations. It serves as another introduction. The first three sections are preparatory. They describe the relevant parts of abstract functoriality in our case.

We also prove the lifting from SO(4) to PGL(4). This amounts to establishing a product of two representations of GL(2) with central characters whose product is 1. Our rigidity theorem for SO(4) amounts to a strong rigidity statement for a pair of representations of $\mathrm{GL}(2,\mathbb{A})$; see [F6]. Our method uses the global tool of the trace formula and the local tool of the fundamental lemma. We deal with all, not only generic or tempered, representations.

For applications to decomposition of cohomology of Shimura varieties see [F7].

1. Homomorphisms of dual groups

Let $\mathbf{G}$ be the projective general linear group $\mathrm{PGL}(4) = \mathrm{PSL}(4)$ over a number field F. Our initial purpose is to determine the automorphic representations π of $\mathbf{G}(\mathbb{A})$, $\mathbb{A}$ being the ring of adèles of F, which are self-contragredient: $\pi \simeq \check{\pi}$, equivalently ([BZ1]), θ-invariant: $\pi \simeq {}^{\theta}\pi$. Here θ, $\theta(g) = J^{-1}{}^{t}g^{-1}J$, is the involution defined by $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where ${}^{t}g$ denotes the transpose of $g \in \mathbf{G}$, and ${}^{\theta}\pi(g) = \pi(\theta(g))$. According to the principle of functoriality ([Bo], [A]) these automorphic representations are essentially described by representations of the Weil group W_F of F into the dual group $\widehat{G} = \mathrm{SL}(4,\mathbb{C})$ of $\mathbf{G}$ which are $\hat{\theta}$-invariant, namely representations of W_F into centralizers $Z_{\widehat{G}}(\hat{s}\hat{\theta})$ of $\mathrm{Int}(\hat{s})\hat{\theta}$ in $\widehat{G}$. Here $\hat{\theta}$ is the dual involution $\hat{\theta}(\hat{g}) = J^{-1}{}^{t}\hat{g}^{-1}J$, and $\hat{s}$ is a semisimple element in $\widehat{G}$. These centralizers are the duals of the twisted (by $\hat{s}\hat{\theta}$) endoscopic groups ([KS]).

A twisted endoscopic group is called elliptic if its dual is not contained in a proper parabolic subgroup of $\widehat{G}$. Representations of nonelliptic endoscopic groups can be reduced by parabolic induction to known ones of smaller rank groups. For our $\widehat{G}$, up to conjugacy the elliptic twisted endoscopic groups have as duals the symplectic group $\widehat{H} = Z_{\widehat{G}}(\hat{\theta}) = \mathrm{Sp}(2,\mathbb{C})$ and the special orthogonal group $\widehat{C} = Z_{\widehat{G}}(\hat{s}\hat{\theta}) =$ "$\mathrm{SO}(4,\mathbb{C})$" $= \{g \in \mathrm{SL}(4,\mathbb{C}); g\hat{s}J{}^{t}g = \hat{s}J = \left(\begin{smallmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{smallmatrix}\right)\}$, of all $A \otimes B = \left(\begin{smallmatrix} aB & bB \\ cB & dB \end{smallmatrix}\right); (A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), B) \in (\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}))/\mathbb{C}^{\times}$ which satisfy $\det A \cdot \det B = 1$. Here $z \in \mathbb{C}^{\times}$ embeds as (z, z^{-1}), $\hat{s} = \mathrm{diag}(-1,1,-1,1)$ and $\omega = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The group $\widehat{H}$ is the dual group of the simple F-group $\mathbf{H} = \mathrm{PSp}(2) = \mathrm{PGSp}(2)$, the projective group of symplectic similitudes. It is the quotient of the group $\mathrm{GSp}(2)$ of $(g, \boldsymbol{\lambda}) \in \mathrm{GL}(4) \times \mathbb{G}_m$ with ${}^{t}gJg = \boldsymbol{\lambda}J$, by its center $\{(\boldsymbol{\lambda}, \boldsymbol{\lambda}^2)\} \simeq \mathbb{G}_m$. Since $\boldsymbol{\lambda}$ is uniquely determined by g (we write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(g)$), we view $\mathrm{GSp}(2)$ as a subgroup of $\mathrm{GL}(4)$ and $\mathrm{PGSp}(2)$ of $\mathrm{PGL}(4)$.

The group $\widehat{C}$ is the dual group of the special orthogonal group $\mathbf{C} =$"SO(4)" of pairs $(g_1, g_2) \in (\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathbb{G}_m$ with $\det g_1 = \det g_2$. Here $z \in \mathbb{G}_m$ embeds as the central element (z, z). Also we write $((\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1))'$ for $\mathbf{C}$, where the prime indicates that the two factors in GL(2) have equal determinants, and $(\dots)''$ if their product is 1.

The principle of functoriality suggests that automorphic discrete spectrum representations of $\mathbf{H}(\mathbb{A})$ and $\mathbf{C}(\mathbb{A})$ parametrize (or lift to) the θ-invariant automorphic discrete spectrum representations of the group $\mathbf{G}(\mathbb{A})$ of $\mathbb{A}$-valued points of $\mathbf{G}$. Our main purpose is to describe this parametrization, in particular define tensor products of two automorphic forms of $\mathrm{GL}(2,\mathbb{A})$ the product of whose central characters is 1, and especially describe the automorphic representations of the projective symplectic group of similitudes of rank two, $\mathrm{PGSp}(2,\mathbb{A})$, in terms of θ-invariant representations of $\mathrm{PGL}(4,\mathbb{A})$.

2. Unramified lifting

We proceed to explain how the liftings are defined, first for unramified representations.

An irreducible admissible representation π of an adèle group $\mathbf{G}(\mathbb{A})$ is the restricted tensor product $\otimes\pi_v$ of irreducible admissible ([BZ1]) representations π_v of the groups $\mathbf{G}(F_v)$ of F_v-points of $\mathbf{G}$, where F_v is the completion of F at the place v of F. Almost all the local components π_v are unramified, that is contain a (unique up to a scalar multiple) nonzero K_v-fixed vector. Here K_v is the standard maximal compact subgroup of $\mathbf{G}(F_v)$, namely the group $\mathbf{G}(R_v)$ of R_v-points, R_v being the ring of integers of the nonarchimedean local field F_v; $\mathbf{G}$ is defined over R_v at almost all v. For such v, an irreducible unramified $\mathbf{G}(F_v)$-module π_v is the unique unramified irreducible constituent in an unramified principal series representation $I(\eta_v)$, normalizedly induced ([BZ2]) from an unramified character η_v of the maximal torus $\mathbf{T}(F_v)$ of a Borel subgroup $\mathbf{B}(F_v)$ of $\mathbf{G}(F_v)$ (extended trivially to the unipotent radical $\mathbf{N}(F_v)$ of $\mathbf{B}(F_v)$). The space of $I(\eta_v)$ consists of the smooth functions $\phi : \mathbf{G}(F_v) \to \mathbb{C}$ with $\phi(ank) = (\delta_v^{1/2}\eta_v)(a)\phi(k)$, $k \in K_v$, $n \in \mathbf{N}(F_v)$, $a \in \mathbf{T}(F_v)$, $\delta_v(a) = \det[\mathrm{Ad}(a)|\,\mathrm{Lie}\,\mathbf{N}(F_v)]$, and the $\mathbf{G}(F_v)$-action is $(g\cdot\phi)(h) = \phi(hg)$, $g, h \in \mathbf{G}(F_v)$.

The character η_v is unramified, thus it factorizes as $\eta_v : \mathbf{T}(F_v)/\mathbf{T}(R_v) \to \mathbb{C}^\times$. As $\mathbf{T}(F_v)/\mathbf{T}(R_v) \simeq X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$, η_v lies in $\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times)$, where $\widehat{T}$ is the maximal torus in the Borel subgroup $\widehat{B}$ of $\widehat{G}$, both fixed in the definition of the (complex) dual group $\widehat{G}$ ([Bo], [Ko]). Now $\mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times) = X_*(\widehat{T}) \otimes \mathbb{C}^\times = \widehat{T} \subset \widehat{G}$, thus the unramified irreducible $\mathbf{G}(F_v)$-module π_v determines a conjugacy class $t(\pi_v) = t(I(\eta_v))$ (the "Langlands parameter") in $\widehat{G}$, represented by the image of η_v in $\widehat{T}$.

For lack of space here we shall describe elsewhere (see [F6]) our results on our secondary lifting λ_1, to $\mathbf{G} = \mathrm{PGL}(4)$ from $\mathbf{C} = \mathrm{SO}(4) = ((\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1))'$.

3. The lifting λ from PGSp(2) to PGL(4)

We now turn to the study of our main lifting λ, and of the automorphic representations of the F-group $\mathbf{H} = \mathrm{PGSp}(2) = \mathrm{GSp}(2)/\mathbb{G}_m$. The center $\mathbb{G}_m$ of $\mathrm{GSp}(2) = \{g \in \mathrm{GL}(4);\ {}^tgJg = \boldsymbol{\lambda}J,\ \exists\boldsymbol{\lambda} = \boldsymbol{\lambda}(g) \in \mathbb{G}_m\}$ consists of the scalar matrices. Its dual group is $\widehat{H} = \mathrm{Sp}(2,\mathbb{C}) = Z_{\widehat{G}}(\widehat{\theta}) \subset \widehat{G} = \mathrm{SL}(4,\mathbb{C})$, where $\widehat{\theta}(g) = J^{-1}{}^tg^{-1}J$.

It has a single elliptic endoscopic group $\mathbf{C}_0$ different from $\mathbf{H}$ itself. Thus

$$\widehat{C}_0 = Z_{\widehat{H}}(\widehat{s}_0) = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d \end{pmatrix} \in \widehat{H} \right\} \simeq \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}),$$

where $\widehat{s}_0 = \mathrm{diag}(-1,1,1,-1)$, and $\mathbf{C}_0 = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$. Write λ_0 for the embedding $\widehat{C}_0 \hookrightarrow \widehat{H}$, and λ for the embedding $\widehat{H} \hookrightarrow \widehat{G}$.

The embedding $\lambda_0 : \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \widehat{H} = \mathrm{Sp}(2,\mathbb{C})$ defines the "endoscopic" lifting $\lambda_0 : \pi_2(\mu_1,\mu_1^{-1}) \times \pi_2(\mu_2,\mu_2^{-1}) \mapsto \pi_{\mathrm{PGSp}(2)}(\mu_1,\mu_2)$. Here $\pi_2(\mu_i,\mu_i^{-1})$ is the unramified irreducible constituent of the normalizedly induced $\mathrm{PGL}(2,F_v)$-module $I(\mu_i,\mu_i^{-1})$ (μ_i are unramified characters of $F_v^\times$, $i=1,2$); $\pi_{\mathrm{PGSp}(2)}(\mu_1,\mu_2)$ is the unramified irreducible constituent of the $\mathrm{PGSp}(2,F_v)$-module $I_{\mathrm{PGSp}(2)}(\mu_1,\mu_2)$ normalizedly induced from the character $n \cdot \mathrm{diag}(\alpha,\beta,\gamma,\delta) \mapsto \mu_1(\alpha/\gamma)\mu_2(\alpha/\beta)$ of the upper triangular subgroup of $\mathrm{PGSp}(2,F_v)$ (n is in the unipotent radical, $\alpha\delta = \beta\gamma$).

The embedding $\lambda : \widehat{H} = \mathrm{Sp}(2,\mathbb{C}) \hookrightarrow \mathrm{SL}(4,\mathbb{C}) = \widehat{G}$ defines the lifting λ which maps the unramified irreducible $\mathrm{PGSp}(2,F_v)$-module $\pi_{\mathrm{PGSp}(2)}(\mu_1,\mu_2)$ to the unramified irreducible $\mathrm{PGL}(4,F_v)$-module $\pi_4(\mu_1,\mu_2,\mu_2^{-1},\mu_1^{-1})$.

The composition $\lambda \circ \lambda_0 : \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \widehat{G} = \mathrm{SL}(4,\mathbb{C})$ takes $\pi_2(\mu_1,\mu_1^{-1}) \times \pi_2(\mu_2,\mu_2^{-1})$ to $\pi_4(\mu_1,\mu_2,\mu_2^{-1},\mu_1^{-1}) = \pi_4(\mu_1,\mu_1^{-1},\mu_2,\mu_2^{-1})$, namely the unramified irreducible $\mathrm{PGL}(2,F_v) \times \mathrm{PGL}(2,F_v)$-module $\pi_2 \times \pi_2'$ to the unramified irreducible constituent $\pi_4(\pi_2,\pi_2')$ of the $\mathrm{PGL}(4,F_v)$-module $I_4(\pi_2,\pi_2')$ normalizedly induced from the representation $\pi_2 \otimes \pi_2'$ of the parabolic of type $(2,2)$ of $\mathrm{PGL}(4,F_v)$ (extended trivially to the unipotent radical). For example $\lambda \circ \lambda_0$ takes the trivial $\mathrm{PGL}(2,F_v) \times \mathrm{PGL}(2,F_v)$-module $\mathbf{1}_2 \times \mathbf{1}_2$ to the unramified irreducible constituent $\pi_4(\mathbf{1}_2,\mathbf{1}_2)$ of $I_4(\mathbf{1}_2,\mathbf{1}_2)$, and $\mathbf{1}_2 \times \pi_2$ to $\pi_4(\mathbf{1}_2,\pi_2) = \pi_4(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$. This last π_4 is traditionally denoted by J.

The definition of lifting is extended from the case of unramified representations to that of any admissible representations. For this purpose we defined in [F5] norm maps from the θ-stable θ-regular conjugacy classes in $G = \mathbf{G}(F)$ to stable conjugacy classes in $H = \mathbf{H}(F)$, and from these to conjugacy classes in $\mathbf{C}_0(F)$. These norm maps extend the norm maps on the split tori in these groups. The latter maps are dual to the homomorphisms λ and λ_0 of the dual groups. This is used to define a relation of matching functions f, f_H and f_{C_0} (they have suitably defined matching orbital integrals) and a dual relation of liftings of representations.

To express the lifting results we use the following notations for induced representations of $H = \mathrm{PGSp}(2,F)$. For characters μ_1, μ_2, σ of $F^\times$ with $\mu_1\mu_2\sigma^2 = 1$ we write $\mu_1 \times \mu_2 \rtimes \sigma$ for the H-module normalizedly induced from the character $mu \mapsto \mu_1(a)\mu_2(b)\sigma(\boldsymbol{\lambda})$, $u \in U$, $m = \mathrm{diag}(a,b,\boldsymbol{\lambda}/b,\boldsymbol{\lambda}/a)$, $a,b,\boldsymbol{\lambda} \in F^\times$, of the upper triangular minimal parabolic of H.

For a $\mathrm{GL}(2,F)$-module π_2 and character μ we write $\pi_2 \rtimes \mu$ for the $\mathrm{PGSp}(2,F)$-module normalizedly induced from the representation $p = mu \mapsto \pi_2(g)\mu(\boldsymbol{\lambda})$, $m = \mathrm{diag}(g, \boldsymbol{\lambda} w^t g^{-1} w)$, $u \in U_{(2)}$, $\boldsymbol{\lambda} \in F^\times$ (here the product of the central character ω of π_2 with μ^2 is 1) of the Siegel parabolic subgroup (whose unipotent radical $U_{(2)}$ is abelian).

We write $\mu \rtimes \pi_2$, if $\omega\mu = 1$, for the representation of $\mathrm{PGSp}(2,F)$ normalizedly induced from the representation $p = mu \mapsto \mu(a)\pi_2(g)$, $m = \mathrm{diag}(a, g, \boldsymbol{\lambda}(g)/a)$,

$u \in U_{(1)}$, $\boldsymbol{\lambda}(g) = \det g$, of the Heisenberg parabolic subgroup (whose unipotent radical $U_{(1)}$ is a Heisenberg group).

These inductions are normalized by multiplying the inducing representation by the character $p \mapsto |\det(\mathrm{Ad}(p))|\,\mathrm{Lie}\,U|^{1/2}$, as usual. For example, $I_H(\mu_1, \mu_2) = \mu_1\mu_2 \times \mu_1/\mu_2 \rtimes \mu_1^{-1}$. Note that $\pi \rtimes \sigma \simeq \check{\pi} \rtimes \omega\sigma$ and $\mu(\pi \rtimes \sigma) = \pi \rtimes \mu\sigma$. Complete results describing reducibility of these induced representations are recorded in [ST]—whose results and notations we use—following earlier work of [Ro2], [Sh2], [Sh3] and [W]. Our lifting results explain the results of [ST].

For properly induced representations, defining λ- and λ_0-liftings by character relations ($\lambda(\pi_H) = \pi_4$ if $\mathrm{tr}\,\pi_4(f \times \theta) = \mathrm{tr}\,\pi_H(f_H)$ for all matching f, f_H, and $\lambda_0(\pi_1 \times \pi_2) = \pi_H$ if $\mathrm{tr}\,\pi_H(f_H) = \mathrm{tr}(\pi_1 \times \pi_2)(f_{C_0})$ for all matching f_H, f_{C_0}), our preliminary results (obtained by local character evaluations) are that $\omega^{-1} \rtimes \pi_2$ λ-lifts to $\pi_4 = I_G(\pi_2, \check{\pi}_2)$, that $\mu\pi_2 \rtimes \mu^{-1}$ (here $\omega = 1$) λ-lifts to $\pi_4 = I_G(\mu, \pi_2, \mu^{-1})$, and that $I_2(\mu, \mu^{-1}) \times \pi_2$ λ_0-lifts from C_0 to $\mu\pi_2 \rtimes \mu^{-1}$ on $H = \mathrm{PGSp}(2, F)$.

Let χ be a character of $F^\times/F^{\times 2}$. It defines a one-dimensional representation $\chi_H(h) = \chi(\boldsymbol{\lambda}(h))$ of H, which λ-lifts to the one-dimensional representation $\chi(g) = \chi(\det g)$ of G (if $h = Ng$ then $\boldsymbol{\lambda}(h) = \det g$; on diagonal matrices $N(\mathrm{diag}(a,b,c,d)) = \mathrm{diag}(ab, ac, db, dc)$). The Steinberg representation of H λ-lifts to the Steinberg representation of G, and for any character χ of $F^\times$ with $\chi^2 = 1$ we have $\lambda(\chi_H\,\mathrm{St}_H) = \chi\,\mathrm{St}_G$.

4. Elliptic representations

Our local liftings, λ, λ_0 and λ_1, are relations of representations, defined by means of character relations. Thus our finer local lifting results concern elliptic representations (whose characters are nonzero on the set of elliptic elements). They follow on using global techniques. Elliptic representations include the cuspidal ones (terminology of [BZ1], [BZ2]; these are called "supercuspidal" by Harish-Chandra, who used the word "cuspidal" for what is currently named "discrete series" or "square integrable" representations).

Local Theorem (PGSp(2) to PGL(4)). *For any unordered pair π_1, π_2 of square integrable irreducible representations of* $\mathrm{PGL}(2,F)$ *there exists a unique pair π_H^+, π_H^- of tempered (square integrable if $\pi_1 \neq \pi_2$, cuspidal if $\pi_1 \neq \pi_2$ are cuspidal) representations of H with*

$$\mathrm{tr}(\pi_1 \times \pi_2)(f_{C_0}) = \mathrm{tr}\,\pi_H^+(f_H) - \mathrm{tr}\,\pi_H^-(f_H),$$
$$\mathrm{tr}\,I_G(\pi_1, \pi_2; f \times \theta) = \mathrm{tr}\,\pi_H^+(f_H) + \mathrm{tr}\,\pi_H^-(f_H)$$

for all triples (f, f_H, f_{C_0}) of matching functions.

If $\pi_1 = \pi_2$ is cuspidal, π_H^+ and π_H^- are the two inequivalent constituents of $1 \rtimes \pi_1$.

If $\pi_1 = \pi_2 = \sigma\,\mathrm{sp}_2$ where σ is a character of $F^\times$ with $\sigma^2 = 1$, then π_H^+ and π_H^- are the two tempered inequivalent constituents $\tau(\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$ and $\tau(\nu^{1/2}\mathbf{1}_2, \sigma\nu^{-1/2})$ of $1 \rtimes \sigma\,\mathrm{sp}_2$.

If $\pi_1 = \sigma\,\mathrm{sp}_2$, $\sigma^2 = 1$, and π_2 is cuspidal, then π_H^+ is the square integrable constituent $\delta(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$ of the induced $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$; π_H^- is cuspidal, denoted here by $\delta^-(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$.

If $\pi_1 = \sigma\,\mathrm{sp}_2$ and $\pi_2 = \xi\sigma\,\mathrm{sp}_2$, ξ $(\neq 1 = \xi^2)$ and σ $(\sigma^2 = 1)$ are characters of $F^\times$, then π_H^+ is the square integrable constituent $\delta(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$ of the induced $\xi\nu^{1/2}\,\mathrm{sp}_2 \rtimes \sigma\nu^{-1/2}$; π_H^- is cuspidal, denoted here by $\delta^-(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$.

For every character σ of $F^\times/F^{\times 2}$ and square integrable π_2 there exists a nontempered representation $\pi_H^\times$ of H such that for all matching f, f_H, f_{C_0}

$$\operatorname{tr}(\sigma\mathbf{1}_2\times\pi_2)(f_{C_0}) = \operatorname{tr}\pi_H^\times(f_H)+\operatorname{tr}\pi_H^-(f_H),$$

$$\operatorname{tr} I_G(\sigma\mathbf{1}_2,\pi_2;f\times\theta) = \operatorname{tr}\pi_H^\times(f_H)-\operatorname{tr}\pi_H^-(f_H).$$

Here $\pi_H^- = \pi_H^-(\sigma\,\mathrm{sp}_2\times\pi_2)$ and $\pi_H^\times = L(\sigma\nu^{1/2}\pi_2,\sigma\nu^{-1/2})$.

For any characters ξ, σ of $F^\times/F^{\times 2}$ and matching f, f_H, f_{C_0} we have

$$\operatorname{tr}(\sigma\xi\mathbf{1}_2\times\sigma\mathbf{1}_2)(f_{C_0}) = \operatorname{tr} L(\nu\xi,\xi\rtimes\sigma\nu^{-1/2})(f_H) - \operatorname{tr} X(\xi\nu^{1/2}\,\mathrm{sp}_2,\xi\sigma\nu^{-1/2})(f_H),$$

$$\operatorname{tr} I_G(\sigma\xi\mathbf{1}_2,\sigma\mathbf{1}_2;f\times\theta) = \operatorname{tr} L(\nu\xi,\xi\rtimes\sigma\nu^{-1/2})(f_H) + \operatorname{tr} X(\xi\nu^{1/2}\,\mathrm{sp}_2,\xi\sigma\nu^{-1/2})(f_H).$$

Here $X=\delta^-$ if $\xi\neq 1$ and $X=L$ if $\xi=1$.

Any θ-invariant irreducible square integrable representation π of G which is not a λ_1-lift is a λ-lift of an irreducible square integrable representation π_H of H, thus $\operatorname{tr}\pi(f\times\theta) = \operatorname{tr}\pi_H(f_H)$ for all matching f, f_H. In particular, the square integrable (resp. nontempered) constituent $\delta(\xi\nu,\nu^{-1/2}\pi_2)$ (resp. $L(\xi\nu,\nu^{-1/2}\pi_2)$) of the induced representation $\xi\nu\rtimes\nu^{-1/2}\pi_2$ of H, where π_2 is a cuspidal (irreducible) representation of $\mathrm{GL}(2,F)$ with central character $\xi\neq 1=\xi^2$ and $\xi\pi_2=\pi_2$, λ-lifts to the square integrable (resp. nontempered) constituent $S(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$ (resp. $J(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$) of the induced representation $I_G(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$ of $G=\mathrm{PGL}(4,F)$. □

These character relations permit us to introduce the notion of a packet of an irreducible representation, and of a quasi-packet, over a local field. Thus we say that the *packet* of a representation π_H of H consists of π_H alone unless it is tempered of the form π_H^+ or π_H^- for some pair π_1, π_2 of (irreducible) square integrable representations of $\mathrm{PGL}(2,F)$, in which case the packet $\{\pi_H\}$ is defined to be $\{\pi_H^+,\pi_H^-\}$, and we write $\lambda_0(\pi_1\times\pi_2)=\{\pi_H^+,\pi_H^-\}$ and $\lambda(\{\pi_H^+,\pi_H^-\}) = I_G(\pi_1,\pi_2)$. Further, we define a *quasi-packet* only for the nontempered (irreducible) representations $\pi_H^\times$ and $L=L(\nu\xi,\xi\rtimes\sigma\nu^{-1/2})$, to consist of $\{\pi_H^\times,\pi_H^-\}$ and $\{L,X\}$, $X=X(\xi\nu^{1/2}\,\mathrm{sp}_2,\xi\sigma\nu^{-1/2})$. We say that $\sigma\mathbf{1}_2\times\pi_2$ λ_0-lifts to the quasi-packet $\lambda_0(\sigma\mathbf{1}_2\times\pi_2)=\{\pi_H^\times,\pi_H^-\}$, which in turn λ-lifts to $I_G(\sigma\mathbf{1}_2,\pi_2)$, and similarly, $\sigma\xi\mathbf{1}_2\times\sigma\mathbf{1}_2$ λ_0-lifts to $\lambda_0(\sigma\xi\mathbf{1}_2\times\sigma\mathbf{1}_2)=\{L,X\}$ which λ-lifts to $I_G(\sigma\xi\mathbf{1}_2,\sigma\mathbf{1}_2)$.

5. Automorphic representations

With these local definitions we can state our global results. These global results are partial, since we work with test functions whose components are elliptic at least at two places. This constraint will be removed once the trace formulae identity is established for general test functions. With further work, known techniques can reduce the constraint to a single place.

With this reservation, emphasized by a $*$-superscript in the following Global Theorem, the discrete spectrum representations of $\mathbf{H}(\mathbb{A})=\mathrm{PGSp}(2,\mathbb{A})$ can now be described by means of the liftings. They consist of two types, stable and unstable. Global packets and quasi-packets define a partition of the spectrum. To define a (global) [quasi-]packet $\{\pi_H\}$, fix a local [quasi-]packet $\{\pi_{Hv}\}$ at each place v of F, such that $\{\pi_{Hv}\}$ contains an unramified member π_{Hv}^0 (and then $\{\pi_{Hv}\}$ consists only of π_{Hv}^0 in case it is a packet) for almost all v. The [quasi-]*packet* $\{\pi_H\}$ is then defined to consist of all products $\otimes_v\pi'_{Hv}$ with π'_{Hv} in $\{\pi_{Hv}\}$ for all v, and $\pi'_{Hv}=\pi_{Hv}^0$ for almost all v. The [quasi-]packet $\{\pi_H\}$ of an automorphic representation π_H is defined by the local [quasi-]packets $\{\pi_{Hv}\}$ of the components π_{Hv} of π_H at almost all places.

The discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ will be described by means of the λ_0- and λ-liftings. We say that the discrete spectrum representation $\pi_1\times\pi_2$ λ_0-*lifts* to a packet $\{\pi_H\}$ (or to a member thereof) if $\{\pi_{Hv}\}=\lambda_0(\pi_{1v}\times\pi_{2v})$ for almost all v, and that a packet $\{\pi_H\}$ (or a member of it) λ-*lifts* to an irreducible self-contragredient automorphic representation π if $\lambda(\{\pi_{Hv}\})=\pi_v$ for almost all v. The *unstable* spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ is defined to be the set of discrete spectrum representations which are λ_0-lifts; its complement is the *stable* spectrum.

Global Theorem* (PGSp(2) to PGL(4)). *The packets and quasi-packets partition the discrete spectrum of the group* $\mathrm{PGSp}(2,\mathbb{A})$*, namely they satisfy the rigidity theorem: if* π_H *and* π'_H *are discrete spectrum representations locally equivalent at almost all places, then their packets or quasi-packets are equal.*

The λ*-lifting is a bijection between the set of packets (resp. quasi-packets) of discrete spectrum representations in the stable spectrum (of* $\mathrm{PGSp}(2,\mathbb{A})$*) and the set of self-contragredient discrete spectrum representations of* $\mathrm{PGL}(4,\mathbb{A})$ *which are one dimensional, or cuspidal and not a* λ_1*-lift from* $\mathbf{C}(\mathbb{A})$ *(resp. residual* $J(\nu^{1/2}\pi_2,\nu^{-1/2}\pi_2)$ *where* π_2 *is a cuspidal representation of* $\mathrm{GL}(2,\mathbb{A})$ *with central character* $\xi\neq 1=\xi^2$ *and* $\xi\pi_2=\pi_2$*).*

The λ_0*-lifting is a bijection between the set of pairs of discrete spectrum representations* $\{\pi_1\times\pi_2,\pi_2\times\pi_1;\pi_1\neq\pi_2\}$ *of* $\mathrm{PGL}(2,\mathbb{A})\times\mathrm{PGL}(2,\mathbb{A})$*, and the set of packets and quasi-packets in the unstable spectrum of* $\mathrm{PGSp}(2,\mathbb{A})$*. The* λ*-lifting is a bijection from this last set to the set of automorphic representations* $I_G(\pi_1,\pi_2)$ *of* $\mathrm{PGL}(4,\mathbb{A})$*, normalizedly induced from the discrete spectrum representation* $\pi_1\times\pi_2$ $(\pi_1\neq\pi_2)$ *on the parabolic subgroup with Levi factor of type* $(2,2)$*. If* $\pi_1\times\pi_2$ *is cuspidal, its* λ_0*-lift is a packet; otherwise, a quasi-packet.*

Each member of a stable packet occurs in the discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ *with multiplicity one. The multiplicity* $m(\pi_H)$ *of a member* $\pi_H=\otimes\pi_{Hv}$ *of an unstable [quasi-]packet* $\lambda_0(\pi_1\times\pi_2)$ $(\pi_1\neq\pi_2)$ *is not ("stable", or) constant over the [quasi-]packet. If* $\pi_1\times\pi_2$ *is cuspidal, it is* $m(\pi_H)=\frac{1}{2}(1+(-1)^{n(\pi_H)})$ $(\in\{0,1\})$*, where* $n(\pi_H)$ *is the number of components* π^-_{Hv} *of* π_H *and each* π_H *with* $m(\pi_H)=1$ *is cuspidal (*$n(\pi_H)$ *is bounded by the number of places* v *where both* π_{1v} *and* π_{2v} *are square integrable).*

The multiplicity $m(\pi_H)$ *(in the discrete spectrum of* $\mathrm{PGSp}(2,\mathbb{A})$*) of* $\pi_H=\otimes\pi_{Hv}$ *of a quasi-packet* $\lambda_0(\sigma\mathbf{1}_2\times\pi_2)$*, where* π_2 *is a cuspidal representation of* $\mathrm{PGL}(2,\mathbb{A})$ *and* σ *is a character of* $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}$*, is* $\frac{1}{2}(1+\varepsilon(\sigma\pi_2,\frac{1}{2})(-1)^{n(\pi_H)})$ $(=0$ *or* $1)$*, where* $n(\pi_H)$ *is the number of components* π^-_{Hv} *of* π_H*, and* $\varepsilon(\sigma\pi_2,s)$ *is the usual* ε*-factor which appears in the functional equation of the* L*-function* $L(\pi_2,s)$*. In particular* $\pi^\times_H=\otimes\pi^\times_{Hv}$ $(n(\pi_H)=0)$ *is in the discrete spectrum if and only if* $\varepsilon(\sigma\pi_2,\frac{1}{2})=1$*.*

Finally we have $m(\pi_H)=\frac{1}{2}(1+(-1)^{n(\pi_H)})$ *for* $\pi_H=\otimes\pi_{Hv}$ *in* $\lambda_0(\sigma\xi\mathbf{1}_2\times\sigma\mathbf{1}_2)$ *with* $n(\pi_H)$ *components* $\pi_{Hv}=X_v$*. Here* $\pi_H=\otimes L_v$ $(n(\pi_H)=0)$ *is residual.* □

6. Unstable spectrum

Note that the quasi-packet $\lambda_0(\sigma\xi\mathbf{1}\times\sigma\mathbf{1}_2)$ is defined by the local quasi-packets $\{L_v=L(\nu_v\xi_v,\xi_v\rtimes\sigma_v\nu_v^{-1/2}),\ X_v=X(\xi_v\nu_v^{1/2}\mathrm{sp}_{2v},\xi_v\sigma_v\nu_v^{-1/2})\}$ for every v, where ξ $(\neq 1)$, σ are characters of $\mathbb{A}^\times/F^\times$ with $\xi^2=1=\sigma^2$ and ξ_v, σ_v are their components. When ξ_v, σ_v are unramified, this quasi-packet contains the unramified representation $\pi^0_{Hv} = L_v$. Members of this quasi-packet have been studied by means of the theta correspondence by Howe and Piatetski-Shapiro; see, e.g., [PS1], Theorem 2.5. They attracted interest since they violate the naive generalization of the Ramanujan conjecture, which expects the components of a cuspidal representation to be tempered. (The form of the Ramanujan conjecture which is expected to be true asserts that the components of a cuspidal representation of PGSp(2,$\mathbb{A}$) which λ-lifts to a cuspidal representation of PGL(4,$\mathbb{A}$) are tempered.) Members of this quasi-packet are equivalent at almost all places to the quotient of the properly induced representation $\nu\xi \times \xi \rtimes \sigma\nu^{-1/2}$.

Let π_2 be a cuspidal representation of PGL(2, $\mathbb{A}$) and σ a character of $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}$. The packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ contains the constituent $\pi_H^\times = \otimes_v \pi^\times_{Hv}$ of the representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2} \simeq \sigma\nu^{-1/2}\pi_2 \rtimes \sigma\nu^{1/2}$ properly induced from an automorphic representation, hence it is automorphic by [L]. It is known that $\pi_H^\times$ is residual precisely when $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; hence $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$ in this case.

Let $n(\pi_2)$ denote the number of square integrable components of π_2. The quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ thus consists of $2^{n(\pi_2)}$ (irreducible) representations. If $n(\pi_2) \geq 1$, half of them are in the discrete spectrum, all cuspidal if $L(\sigma\pi_2, \frac{1}{2}) = 0$, and all but one: $\pi_H^\times = \otimes_v \pi^\times_{Hv}$, are cuspidal if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$. If $n(\pi_2) \geq 1$ and $L(\sigma\pi_2, \frac{1}{2}) = 0$, the automorphic nonresidual $\pi_H^\times$ is cuspidal when $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$.

If π_2 has no square integrable components ($n(\pi_2) = 0$), the packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ consists only of $\pi_H^\times$. This $\pi_H^\times$ is residual if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; cuspidal (by [PS1], Theorem 2.6, and [PS2], Theorem A.2) if $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$; or (automorphic but) not in the discrete spectrum otherwise: $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = -1$. In this last case the λ_0-lift of $\sigma\mathbf{1}_2 \times \pi_2$ is not in the discrete spectrum, and there is no discrete spectrum representation λ-lifting to $I_G(\sigma\mathbf{1}_2, \pi_2)$.

At a place v where π_{2v} is induced $I(\mu_v, \mu_v^{-1})$, the packet $\pi_{Hv} = \lambda_0(\sigma_v\mathbf{1}_2 \times \pi_{2v})$ is the irreducible induced $\mu_v\sigma_v\mathbf{1}_2 \rtimes \mu_v^{-1}$, which λ-lifts to $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$, and *not* the irreducible induced

$$\sigma_v\mu_v\nu_v^{1/2} \times \sigma_v\mu_v^{-1}\nu_v^{1/2} \rtimes \sigma_v\nu_v^{-1/2} = \sigma_v\mu_v\nu_v^{1/2} \rtimes I(\mu_v^{-1}, \sigma_v\nu_v^{-1/2}),$$

which λ-lifts to the reducible induced $I_G(\mu_v, \sigma_v I(\nu_v^{1/2}, \nu_v^{-1/2}), \mu_v^{-1})$, which has the constituent $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$.

Members of the quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ were studied numerically by H. Saito and N. Kurokawa, and using the theta correspondence by Piatetski-Shapiro and others; see [PS1], Theorem 2.6. They attracted interest since they violate the naive generalization of the Ramanujan conjecture. They are equivalent at almost all places to the quotient of the properly induced representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$.

A discrete spectrum representation π_H with component $L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$ (whose packet consists of itself), where π_{2v} is a cuspidal representation with central character $\xi_v \neq 1 = \xi_v^2$ and $\xi_v\pi_{2v} = \pi_{2v}$, is in the packet of $L(\nu\xi, \nu^{-1/2}\pi_2)$. Here π_2 is cuspidal with central character $\xi \neq 1 = \xi^2$, hence $\xi\pi_2 = \pi_2$, whose components at v are π_{2v} and ξ_v. It λ-lifts to $J_G(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. At v with $\xi_v = 1$ the component π_{2v} is induced. If $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\xi_v^2 = 1$ and $\mu_v^2 = 1$ (in particular whenever $\xi_v \neq 1$ and π_{2v} is not cuspidal), then $L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$ is $L_v = L(\nu_v\xi_v, \xi_v \rtimes \mu_v\nu_v^{-1/2})$, which λ-lifts to $I_G(\mu_v\mathbf{1}_2, \mu_v\xi_v\mathbf{1}_2)$. Its packet contains also $X_v = X(\nu_v^{1/2}\xi_v\,\mathrm{sp}_{2v}, \xi_v\mu_v\nu_v^{-1/2})$. Thus the packet of π_H is determined by $\{L_v, X_v\}$ at all v where $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\mu_v^2 = 1 = \xi_v^2$, and by the singleton $\{L_v = L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})\}$ at all other v, where

π_{2v} is cuspidal, or $\xi_v = 1$ and $\pi_{2v} = I(\mu_v, \mu_v^{-1})$, $\mu_v^2 \neq 1$. Each member of this infinite packet occurs in the discrete spectrum with multiplicity one, and is cuspidal, with the exception of $L(\nu\xi, \nu^{-1/2}\pi_2) = \otimes_v L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$, which is residual ([Kim], Theorem 7.2). Members of the packet $L(\nu\xi, \nu^{-1/2}\pi_2)$ are considered in the Appendix of [PS1] and its corrigendum.

If π_1 and π_2 are cuspidal but there is no place v where both are square integrable, $\lambda_0(\pi_1 \times \pi_2)$ consists of a single irreducible cuspidal representation. This instance of the lifting λ_0—where π_i are cuspidal—can also be studied ([Rb]) using the theta correspondence for suitable dual reductive pairs (SO(4), PGSp(2)) for the isotropic and anisotropic forms of the orthogonal group, to describe further properties of the packets, such as their periods.

7. Generic representations

Our proof of the existence of the lifting λ uses only the trace formula, orbital integrals and character relations. To compute the multiplicities in the discrete spectrum we use a global proof. It relies on results of [GRS], [KRS], and [Sh1] from the theory of generic representations.

This global proof in our case of PGSp(2) is similar to the second proof of [F4, II], Proposition 3.5, p. 48, in the case of U(3). The 2nd proof of [F4, II] is also based on the theory of generic representations. But it is not complete. Indeed, the claim in Proposition 2.4(i) in reference [GP] to [F4, II] that "$L^2_{0,1}$ has multiplicity 1", is interpreted in [F4, II] as asserting that generic representations of U(3) occur in the discrete spectrum with multiplicity one. But it should be interpreted as asserting that irreducible π in $L^2_{0,1}$ have multiplicity one *only in the subspace* $L^2_{0,1}$ of the discrete spectrum. This assertion does not exclude the possibility of having a cuspidal π' perpendicular and equivalent to $\pi \subset L^2_{0,1}$. Multiplicity one for the generic spectrum of U(3) could be deduced from Proposition 2.4(i) in [GP] of [F4, II] using the statement that a locally generic representation is globally generic. (In fact this statement is equivalent to multiplicity one.)

In our case of PGSp(2) a similar statement follows from [KRS], [GRS], [Sh1].

A proof for U(3) (independent of the local proof of [F4, II], p. 47) still needs to be given.

The first proof of Proposition 3.5 in [F4, II], p. 47, namely that the multiplicity one theorem holds for the discrete spectrum of U(3), is purely local. It uses only character relations. It is based on a twisted analogue of Rodier's result [Ro1] that the number of Whittaker models (0 or 1) is encoded in the behavior of the character near the origin. For further details see [F4, IV]. Such a technique was first used in [FK], to prove the multiplicity one theorem for the metaplectic group of GL(n).

There is some overlap between our results on the existence of the λ-lifting and the work of [GRS]. However, the methods of [GRS] apply only to generic representations, while our methods apply to all representations of PGSp(2). Thus we can define packets, describe their structure, establish the multiplicity one theorem and rigidity theorem for packets of PGSp(2), specify which member in a packet or a quasi-packet is in the discrete spectrum, and we can also λ-lift the nongeneric nontempered (at almost all places) packets to residual self-contragredient representations of PGL($4, \mathbb{A}$). Our liftings are proven in terms of all places, not only almost all places. In addition we establish the lifting λ_1 from SO(4) to PGL(4), determine its fibers (that is, prove the multiplicity one theorem for SO(4) and rigidity in the sense explained in [F6] and [F1]), and show that any self-contragredient discrete spectrum representation of PGL$(4, \mathbb{A})$ which is not a λ-lift from PGSp$(2, \mathbb{A})$ is a λ_1-lift from SO$(4, \mathbb{A})$.

Our work is an analogue for (SO(4), PGSp(2), PGL(4)) of [F3], which dealt with the symmetric square lifting, thus with (PGL(2), SL(2), PGL(3)), and of [F4], which dealt with quadratic base change for the unitary group U$(3,E/F)$, thus with (U$(2,E/F)$, U$(3,E/F)$, GL$(3,E)$). These works use the twisted—by transpose-inverse (and the Galois action in the unitary groups case)—trace formulae on PGL(4), PGL(3), GL$(3,E)$. They are based on the fundamental lemma: [F5] in our case, [F3, V] and [F4, I] in the other cases. The technique employed in these last papers benefited from the work of Weissauer [We] and Kazhdan [K].

The present work, which deals with the applications of the fundamental lemma and the trace formula to character relations, liftings and the definition of packets, is analogous to [F3, IV] and [F4, II]. The trace formulae identity is proven in [F3, VI] and [F4, III] for all test functions. Here we deal only with test functions which have at least two elliptic components. The trace formulae identity for a general test function has not yet been proven in our case. Arthur is working on this. It would be interesting to pursue the elementary techniques of [F3, VI] and [F4, III], and [F2], which establishes the trace formulae identity for base change for GL(2) by elementary means, based on the usage of regular, Iwahori test functions. In particular our work does not develop the trace formula. It only uses a form of it.

Our approach to the lifting uses the trace formula developed by Arthur, as envisaged by Langlands e.g. in his work on base change for GL(2). It is compatible with the conjectures of Arthur [A].

Of course Siegel modular forms have been extensively studied by many authors (e.g. Siegel, Maass, Shimura, Andrianov, and others) over a long period of time.

An important representation theoretic approach alternative to the trace formula, based on the theta correspondence, Weil representation, Howe's dual reductive pairs, L-functions and converse theorems, has been fruitfully developed in our context of the symplectic group by many authors, see, e.g., [PS1], [PS2], [KRS], [GRS].

Another—purely local—approach is developed in [FZ].

References

[A] J. Arthur, On some problems suggested by the trace formula, *Lie group representations* II, Springer Lecture Notes 1041 (1984), 1–49; Unipotent automorphic representations: conjectures, in: Orbites unipotentes et representations, II, *Astérisque* 171–172 (1989), 13–71. MR **85k:**11025; MR **91f:**22030

[BZ1] J. Bernstein, A. Zelevinskii, Representations of the group GL(n, F) where F is a nonarchimedean local field, *Russian Math. Surveys* 31 (1976), 1–68. MR **54:**12988

[BZ2] J. Bernstein, A. Zelevinsky, Induced representations of reductive p-adic groups, *Ann. Sci. École Norm. Sup.* 10 (1977), 441–472. MR **58:**28310

[Bo] A. Borel, Automorphic L-functions, *Proc. Sympos. Pure Math.* 33 II, Amer. Math. Soc., Providence, R.I. (1979), 27–63. MR **81m:**10056

[F1] Y. Flicker, Lifting automorphic forms of PGSp(2) and SO(4) to PGL(4), 2001.

[F2] Y. Flicker, Regular trace formula and base-change lifting, *Amer. J. Math.* 110 (1988), 739–764. MR **89m:**11051

[F3] Y. Flicker, On the symmetric square. IV. Applications of a trace formula, *Trans. Amer. Math. Soc.* 330 (1992), 125–152; V. Unit elements, *Pacific J. Math.* 175 (1996), 507–526; VI. Total global comparison; *J. Funct. Analyse* 122 (1994), 255–278. See also: Automorphic representations of low rank groups, research monograph, 2003. MR **92g:**11052; MR **98a:**11065; MR **96h:**22004

[F4] Y. Flicker, I. Elementary proof of a fundamental lemma for a unitary group, *Canad. J. Math.* 50 (1998), 74–98; II. Packets and liftings for U(3), *J. Analyse Math.* 50 (1988), 19–63; III. Base change trace identity for U(3), *J. Analyse Math.* 52 (1989), 39–52; IV. Characters, genericity, and multiplicity one for U(3), preprint. See also: Automorphic representations of the unitary group U(3,E/F), research monograph, 2003. MR **99e:**22020; MR **90b:**11052; MR **90m:**11080

[F5] Y. Flicker, Matching of orbital integrals on GL(4) and GSp(2), *Mem. Amer. Math. Soc.* no. 655, vol. 137 (1999), 1–114. MR **99g:**11068

[F6] Y. Flicker, Automorphic forms on SO(4), 2004, preprint.

[F7] Y. Flicker, On Zeta functions of Shimura varieties of PGSp(2), 2003, preprint.

[FK] Y. Flicker, D. Kazhdan, Metaplectic correspondence, *Publ. Math. IHES* 64 (1987), 53–110. MR **88d:**11049

[FZ] Y. Flicker, D. Zinoviev, Twisted character of a small representation of PGL(4), *Moscow Math. J.* (2004), in press.

[GRS] D. Ginzburg, S. Rallis, D. Soudry, On Generic Forms on SO($2n+1$); Functorial Lift to GL($2n$), Endoscopy and Base Change, *Internat. Math. Res. Notices* 14 (2001), 729–763. MR **2002g:**11065

[H] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, notes by S. DeBacker and P. Sally, AMS Univ. Lecture Series 16 (1999); see also: *Queen's Papers in Pure and Appl. Math.* 48 (1978), 281–346. MR **2001b:**22015; MR **58:**28313

[JS] H. Jacquet, J. Shalika, On Euler products and the classification of automorphic forms II, *Amer. J. Math.* 103 (1981), 777–815. MR **82m:**10050b

[K] D. Kazhdan, On lifting, in *Lie groups representations II*, Springer Lecture Notes 1041 (1984), 209–249. MR **86h:**22029

[Kim] H. Kim, Residual spectrum of odd orthogonal groups, *Internat. Math. Res. Notices* 17 (2001), 873–906. MR **2002k:**11071

[Ko] R. Kottwitz, Stable trace formula: cuspidal tempered terms, *Duke Math. J.* 51 (1984), 611–650. MR **85m:**11080

[KS] R. Kottwitz, D. Shelstad, Foundations of twisted endoscopy, *Astérisque* 255 (1999), vi+190 pp. MR **2000k:**22024

[KRS] S. Kudla, S. Rallis, D. Soudry, On the degree 5 L-function for Sp(2), *Invent. Math.* 107 (1992), 483–541. MR **93b:**11061

[L] R. Langlands, On the notion of an automorphic representation, *Proc. Sympos. Pure Math.* 33 I (1979), 203–207. MR **81m:**10055

[MW] C. Moeglin, J.-L. Waldspurger, Le spectre résiduel de GL(n), *Ann. Sci. École Norm. Sup.* 22 (1989), 605–674. MR **91b:**22028

[PS1] I. Piatetski-Shapiro, On the Saito-Kurokawa lifting, *Invent. Math.* 71 (1983), 309-338; A remark on my paper: "On the Saito-Kurokawa lifting", *Invent. Math.* 76 (1984), 75–76. MR **84e:**10038; MR **85e:**11038

[PS2] I. Piatetski-Shapiro, Work of Waldspurger, *Lie group representations* II, Springer Lecture Notes 1041 (1984), 280–302. MR **86g:**11030

[Rb] B. Roberts, Global L-packets for GSp(2) and theta lifts, *Doc. Math.* 6 (2001), 247–314. MR **2003a:**11059

[Ro1] F. Rodier, Modèle de Whittaker et caractères de représentations, *Noncommutative harmonic analysis*, Springer Lecture Notes 466 (1975), 151–171. MR **52:**14165

[Ro2] F. Rodier, Sur les représentations non ramifiées des groupes réductifs p-adiques; l'example de GSp(4), *Bull. Soc. Math. France* 116 (1988), 15–42. MR **89i:**22033

[ST] P. Sally, M. Tadic, Induced representations and classifications for GSp($2,F$) and Sp($2,F$), *Mem. Soc. Math. France* 52 (1993), 75–133. MR **94e:**22030

[Sh1] F. Shahidi, On certain L-functions, *Amer. J. Math.* 103 (1981), 297–355. MR **82i:**10030

[Sh2] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.* 132 (1990), 273–330. MR **91m:**11095

[Sh3] F. Shahidi, Langlands' conjecture on Plancherel measures for p-adic groups, in *Harmonic analysis on reductive groups*, 277–295, Progr. Math., 101, Birkhäuser, Boston, MA, 1991. MR **93h:**22033

[W] J.-L. Waldspurger, Un exercise sur $\mathrm{GSp}(4,F)$ et les représentations de Weil, *Bull. Soc. Math. France* 115 (1987), 35–69. MR **89a:**22033

[We] R. Weissauer, A special case of the fundamental lemma, preprint.

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