ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 10, Pages 78–87 (August 3, 2004)
S 1079-6762(04)00132-5

WAVELETS WITH COMPOSITE DILATIONS

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(Communicated by Boris Hasselblatt)

Abstract. A wavelet with composite dilations is a function generating an orthonormal basis or a Parseval frame for $L^2(\mathbb{R}^n)$ under the action of lattice translations and dilations by products of elements drawn from non-commuting matrix sets A and B. Typically, the members of B are shear matrices (all eigenvalues are one), while the members of A are matrices expanding or contracting on a proper subspace of $\mathbb{R}^n$. These wavelets are of interest in applications because of their tendency to produce "long, narrow" window functions well suited to edge detection. In this paper, we discuss the remarkable extent to which the theory of wavelets with composite dilations parallels the theory of classical wavelets, and present several examples of such systems.

1. Introduction

There is considerable interest, both in mathematics and its applications, in the study of efficient representations of multidimensional functions. The motivation comes partly from signal processing, including applications in image compression and feature extraction, and from the investigation of certain classes of singular integral operators. In particular, it was pointed out in several recent research papers that oriented oscillatory waveforms play a fundamental role in the construction of representations for multidimensional functions and signals (cf. [1], [3], [5], and articles in [11]). For example, it was shown that, in order to be optimally sparse in a certain sense, such representations must contain basis elements with many locations, scales, shapes and directions, unlike the trigonometric bases or even the "classical" wavelets, which are made of essentially isotropic oscillatory bumps at various scales and locations (cf. [2]).

In this paper, we introduce a new class of representation systems which have exactly the features we have described, as well as several other properties which have a great potential in applications. These systems, that we call **affine systems with composite dilations**, have the form

$$\Psi_{AB} = \{D_a\, D_b\, T_k\, \psi^\ell :\ k \in \mathbb{Z}^n, b \in B, a \in A, \ell = 1, \dots, L\}, \tag{1.1}$$

Received by the editors February 23, 2004 and, in revised form, April 13, 2004.
2000 *Mathematics Subject Classification.* Primary 42C15, 42C40.
Key words and phrases. Affine systems, frames, multiresolution analysis (MRA), multiwavelets, wavelets.
The fourth author was supported in part by a SW Bell Grant.

where $\psi^\ell \in L^2(\mathbb{R}^n)$, T_k are the **translations** defined by $T_k\, f(x) = f(x-k)$, D_a are the **dilations** defined by $D_a\, f(x) = |\det a|^{-1/2} f(a^{-1}x)$, and A, B are countable subsets of $GL_n(\mathbb{R})$. By choosing ψ^ℓ, A, and B appropriately, we can make Ψ_{AB} an orthonormal (ON) basis or, more generally, a Parseval frame (PF) for $L^2(\mathbb{R}^n)$. In this case, we call $\Psi = \{\psi^1, \ldots, \psi^L\}$ an **ON** AB**-multiwavelet** or a **PF** AB**-multiwavelet**, respectively. If the system has only one generator, that is, $L = 1$, then we use the expression **wavelet** rather than **multiwavelet** in this definition.

As we will show, the mathematical theory of these systems provides a simple and flexible framework for the construction of several classes of bases and Parseval frames. For example, in Sections 3.3 and 3.4, we construct composite wavelets with good time-frequency decay properties, whose elements contain "long and narrow" waveforms with many locations, scales, shapes and directions. These examples have similarities to the *curvelets* [1] and *contourlets* [4], which have been recently introduced in order to obtain efficient representations of natural images. Our approach is more general and presents a simple method for obtaining orthonormal bases and Parseval frames that exhibit these and other geometric features. In particular, our approach extends naturally to higher dimensions and allows a multiresolution construction which is well suited to a fast numerical implementation.

Before embarking on our presentation, it is useful to establish some notation and definitions. Recall that a countable family $\{\psi_i\}_{i\in\mathcal{I}}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **Parseval frame (PF)** for $\mathcal{H}$, if

$$||f||^2 = \sum_{i\in\mathcal{I}} |\langle f, \psi_i\rangle|^2$$

for each $f \in \mathcal{H}$. We adopt the convention that an element $x \in \mathbb{R}^n$ is a column vector, while $\xi \in \widehat{\mathbb{R}}^n$ is a row vector. A vector x multiplying a matrix $a \in GL_n(\mathbb{R})$ on the right, is understood to be a column vector, while a vector ξ multiplying a on the left is a row vector. Thus, $ax \in \mathbb{R}^n$ and $\xi a \in \widehat{\mathbb{R}}^n$. The Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi x}\, dx.$$

For any $E \subset \widehat{\mathbb{R}}^n$, we denote by $L^2(E)^\vee$ the space $\{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset E\}$.

2. AB-MRA

Let B be a countable subset of $\widetilde{SL}_n(\mathbb{Z}) = \{b \in GL_n(\mathbb{R}) : |\det b| = 1\}$ and $A = \{a_i : i \in \mathbb{Z}\}$, where $a_i \in GL_n(\mathbb{R})$. We say that a sequence $\{V_i\}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is an AB**-multiresolution analysis (**AB**-MRA)** if the following holds:

(i) *$D_b\, T_k\, V_0 = V_0$, for any $b \in B$, $k \in \mathbb{Z}^n$.*
(ii) *For each $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$,, where $V_i = D_{a_i}^{-1}\, V_0$.*
(iii) *$\bigcap V_i = \{0\}$ and $\overline{\bigcup V_i} = L^2(\mathbb{R}^n)$.*
(iv) *There exists $\phi \in L^2(\mathbb{R}^n)$ such that $\Phi_B = \{D_b\, T_k\, \phi : b \in B,\ k \in \mathbb{Z}^n\}$ is a semi-orthogonal PF for V_0, that is, Φ_B is a PF for V_0 and, in addition, $D_b\, T_k\, \phi \perp D_{b'}\, T_{k'}\, \phi$ for any $b \neq b'$, $b, b' \in B$, $k, k' \in \mathbb{Z}^n$.*

The space V_0 is called an AB **scaling space** and the function ϕ is an AB **scaling function** for V_0. If, in addition, Φ_B is an orthonormal basis for V_0, then ϕ is an **ON** AB **scaling function**. Also, let W_0 be the orthogonal complement of V_0 in V_1, that is, $V_1 = V_0 \oplus W_0$. We cite the following elementary result [6]:

Theorem 2.1. (i) *Let* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *be such that* $\{D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\}$ *is a PF for* W_0. *Then* Ψ *is a PF AB-multiwavelet.*

(ii) *Let* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *be such that* $\{D_b T_k \psi^\ell : b \in B, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ *is an orthonormal basis for* W_0. *Then* Ψ *is an ON AB-multiwavelet.*

In the situation described by the hypotheses of this theorem (where Ψ is not only a PF for $L^2(\mathbb{R}^n)$, but it is also derived from an AB-MRA), we say that Ψ is a **PF MRA** AB**-multiwavelet** or an **ON MRA** AB**-multiwavelet**, respectively.

In most cases of interest, the set $A \subset GL_n(\mathbb{R})$ has the form $A = \{a_i = a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$, and B is a subgroup of $\widetilde{SL}_n(\mathbb{Z})$ which satisfies $aBa^{-1} \subseteq B$. For the remainder of this section, we will make this assumption.

We say that the PF MRA AB-wavelet ψ is of **finite filter (FF) type** if there exists an AB scaling function ϕ for V_0 and a finite set $\{b_1, \ldots, b_k\} \subset B$ such that

$$\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi)\, \hat{\phi}(\xi\, b_j), \quad \hat{\psi}(\xi a) = \sum_{j=1}^{k} m_1^{(j)}(\xi)\, \hat{\phi}(\xi\, b_j),$$

where $m_0^{(j)}, m_1^{(j)}$, $1 \le j \le k$, are periodic functions. Similarly, the ON MRA AB-multiwavelet Ψ is of finite filter (FF) type if there exists an AB scaling function ϕ for V_0 and a finite set $\{b_1, \ldots, b_k\} \subset B$ such that

$$\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi)\, \hat{\phi}(\xi\, b_j), \quad \hat{\psi}^\ell(\xi a) = \sum_{j=1}^{k} m_{1,\ell}^{(j)}(\xi)\, \hat{\phi}(\xi\, b_j),\ \ell = 1, \ldots, L,$$

where $m_0^{(j)}, m_{1,\ell}^{(j)}$, $1 \le j \le k$, are periodic functions.

It turns out that, while it is possible to construct a PF AB-wavelet using a single generator, that is, $\Psi = \{\psi\}$, in the case of *orthonormal* MRA AB-multiwavelets, multiple generators are needed, that is, $\Psi = \{\psi^1, \ldots, \psi^L\}$, where $L > 1$. This situation is similar to the classical MRA and, as in that case, such restriction is not needed if the system does not come from an MRA (cf., for example, [12]). We refer to [6] for a proof of the following theorems.

Theorem 2.2. *Let* $\Psi = \{\psi^1, \ldots, \psi^L\}$ *be an ON MRA AB-multiwavelet for* $L^2(\mathbb{R}^n)$, *and let* $N = |B/aBa^{-1}|$ (= *the order of the quotient group* B/aBa^{-1}). *Assume that* $|\det a| \in \mathbb{N}$. *Then* $L = N\,|\det a| - 1$.

Theorem 2.3. *Let* $B \subset \widetilde{SL}_n(\mathbb{Z})$ *and* $a \in GL_n(\mathbb{Z})$ *with* $aBa^{-1} \subseteq B$. *Let* $L = N\,|\det a| - 1$, *where* $N = |B/aBa^{-1}|$. *Assume that* $\phi \in L^2(\mathbb{R}^n)$ *is an ON AB scaling function for* $V_0 = \overline{\text{span}\,\{D_b T_k \phi : k \in \mathbb{Z}^n, b \in B\}}$. *Then:*

(i) *There exist ON MRA AB-multiwavelets* $\Psi = \{\psi^1, \ldots, \psi^L\}$ *with scaling space* V_0.

(ii) *If* $\hat{\phi} = \chi_U$, *where* $U \subset \mathbb{R}^n$ *is a measurable set, then there are sets* $T_\ell \subset \mathbb{R}^n$, $\ell = 1, \ldots, L$, *for which* $\Psi = \{\psi^\ell = (\chi_{T_\ell})^\vee : \ell = 1, \ldots, L\}$ *is an ON MRA AB-multiwavelet and* Ψ *is of FF type.*

Remark 2.4. Let $a \in GL_n(\mathbb{Z})$. Under additional assumptions on B (which are satisfied, for example, by the examples in Section 3), there exist sets $S \subset \widehat{\mathbb{R}}^n$ such that ψ is an ON AB-wavelet for $L^2(\mathbb{R}^n)$, where $\hat{\psi} = \chi_S$. It is clear, by Theorem 2.2, that these AB-wavelets are **not** of MRA type.

3. Examples

This section shows that there are several examples of affine systems with composite dilations forming ON bases or PFs for $L^2(\mathbb{R}^n)$. In particular, Sections 3.1 and 3.2 contain examples of PF MRA AB-multiwavelets and ON MRA AB-multiwavelets for $L^2(\mathbb{R}^2)$, respectively; Sections 3.3 and 3.4 describe how to construct PF AB-wavelets which are well-localized both in $\mathbb{R}^2$ and $\widehat{\mathbb{R}}^2$; Section 3.5 describes generalizations of these examples for dimensions $n > 2$; finally, Section 3.6 describes a class of singly generated (non-MRA) ON AB-wavelets. More details about these constructions are found in [6].

3.1. **Example 1.** Let $B = \{b_j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) : j \in \mathbb{Z}\}$. For $0 \le \alpha < \beta$, let $S(\alpha,\beta) = \{\xi = (\xi_1,\xi_2) \in \widehat{\mathbb{R}}^2 : \alpha \le |\xi_1| < \beta\}$ and let $T(\alpha,\beta) = T^+(\alpha,\beta) \cup T^-(\alpha,\beta)$, where $T^+(\alpha,\beta)$ is the trapezoid with vertices $(\alpha,0)$, (α,α), $(0,\beta)$ and (β,β), and $T^-(\alpha,\beta) = \{\xi \in \widehat{\mathbb{R}}^2 : -\xi \in T^+(\alpha,\beta)\}$. Observe that $T(0,\beta)$ is the union of two triangles. A simple computation shows that $T(\alpha,\beta)$ is a **B-tiling region** for $S(\alpha,\beta)$, that is, $S(\alpha,\beta)$ is the disjoint union, modulo null sets, of the sets $T(\alpha,\beta)\, b$, $b \in B$.

Now let $a = \left(\begin{smallmatrix} c & d \\ 0 & e \end{smallmatrix}\right) \in GL_2(\mathbb{R})$, with $|c| > 1$, $S_0 = S(0,1/2|c|)$, $S_i = S_0\, a^i$, $i \in \mathbb{Z}$, and define a sequence $\{V_i\}_{i\in\mathbb{Z}}$ of nested subspaces of $L^2(\mathbb{R}^2)$ by $V_i = L^2(S_i)^\vee$, $i \in \mathbb{Z}$. Also, let $U = T(0,(2|c|)^{-1})$ and $\phi = (\chi_U)^\vee$. Since U is a B-tiling region for S_0, it follows that $\overline{\operatorname{span}\{D_b\, T_k\, \phi : k \in \mathbb{Z}^2, b \in B\}} = L^2(S_0)^\vee = V_0$. This shows that the spaces $\{V_i\}$ form an AB-MRA and ϕ is the AB scaling function for V_0. In addition, for $R = T((2|c|)^{-1}, 2^{-1})$, the function $\psi = (\chi_R)^\vee$ is a PF MRA AB-wavelet, since $\{D_b\, T_k\, \psi : b \in B, k \in \mathbb{Z}^2\}$ is a PF for $W_0 = L^2(S_1 \setminus S_0)^\vee$. Observe that W_0 is the orthogonal complement of V_0 in V_1. Also, ψ is of FF type, since $U\, a^{-1}$ and $R\, a^{-1}$ are contained in $\bigcup_{j=0}^{k-1} U b_j$ for $k \ge |c|$.

3.2. **Example 2.** Let B, $S(\alpha,\beta)$, and $T(\alpha,\beta)$ be defined as in Example 1. Let $a = \left(\begin{smallmatrix} k_1 & k_2 \\ 0 & k_3 \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, with $k_1 > 1$ and $N = |k_1/k_3| \in \mathbb{N}$, and let $L = |\det a| N - 1 = k_1^2 - 1$. Similarly to Example 1, let $S_0 = S(0,1)$, $S_i = S_0\, a^i$, $i \in \mathbb{Z}$, and define a sequence of spaces $\{V_i\}_{i\in\mathbb{Z}}$ by $V_i = L^2(S_i)^\vee$, $i \in \mathbb{Z}$. Let $U = T(0,1)$ and $\phi = (\chi_U)^\vee$. Then ϕ is an ON basis for $\overline{\operatorname{span}\{D_b\, T_k\, \phi : k \in \mathbb{Z}^n, b \in B\}} = L^2(S_0)^\vee = V_0$, and the spaces $\{V_i\}_{i\in\mathbb{Z}}$ form an AB-MRA, where ϕ is the corresponding ON AB scaling function.

Observe that the set $R = T(1,k_1)$ is a B-tiling region for $S_1 \setminus S_0$. We can pick disjoint subsets R_ℓ, $\ell = 1,\dots,L$ of R such that $R = \bigcup_{\ell=1}^{L} R_\ell$ and, for each ℓ, R_ℓ is a fundamental domain for $\mathbb{Z}^2$. Figure 1 illustrates this construction in the special case where $k_1 = 1$ and, thus, $L = 3$. Now let $\Psi = \{\psi^1,\dots,\psi^L\}$, where $\psi^\ell = (\chi_{R_\ell})^\vee$, $\ell = 1,\dots,L$. Since $\{D_b\, T_k\, \psi^\ell : b \in B, \ell = 1,\dots,L, k \in \mathbb{Z}^2\}$ is an ON basis for $W_0 = L^2(S_1 \setminus S_0)^\vee$, and W_0 is the orthogonal complement of V_0 in V_1, it follows that Ψ is an ON MRA AB-multiwavelet. In addition, Ψ is of FF type, since the sets $U\, a^{-1}$ and $R_\ell\, a^{-1}$, $\ell = 1,\dots,L$, are contained in $\bigcup_{j=0}^{k_1-1} U\, b_j$.

The examples described in Sections 3.1 and 3.2 are well-localized in $\widehat{\mathbb{R}}^n$, since the AB-multiwavelets that we constructed are band-limited, but they do not have good localization in $\mathbb{R}^n$, since they decay only as fast as $|x|^{-1}$ when $|x| \to \infty$. The examples described in the next two sections, on the contrary, are well-localized both in $\mathbb{R}^n$ and $\widehat{\mathbb{R}}^n$.

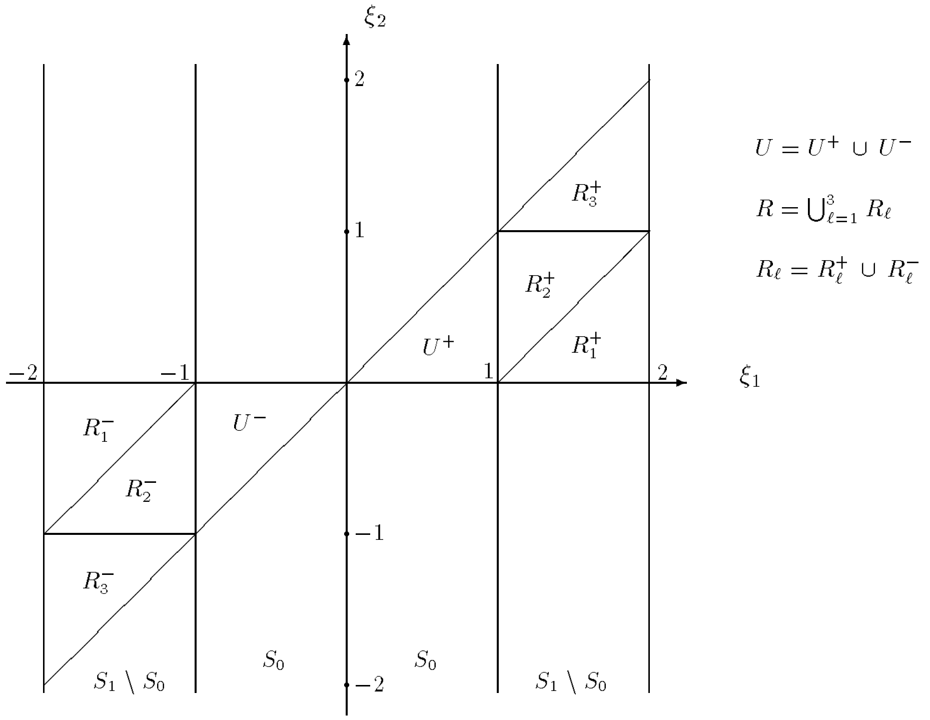

FIGURE 1. The sets U, R and R_ℓ, $\ell = 1, 2, 3$, in Example 2, where $L = 3$.

3.3. **Example 3.** Let $\psi_1 \in L^2(\mathbb{R})$ be a (one-dimensional) dyadic band-limited wavelet with $\operatorname{supp} \hat{\psi}_1 \subset [-\Omega, \Omega]$, $\Omega > 0$, and $\psi_2 \in L^2(\mathbb{R})$ be another band-limited function with $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$ and satisfying

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}_2(\xi + j)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{3.1}$$

As we will show later on, there are several choices of functions ψ_1 and ψ_2 satisfying these properties.

For any $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$, $\omega_1 \neq 0$, define $\psi \in L^2(\mathbb{R}^2)$ by

$$\hat{\psi}(\omega) = \hat{\psi}_1(2^s\, \omega_1)\, \hat{\psi}_2\Big(\frac{\omega_2}{\omega_1}\Big), \tag{3.2}$$

where $s \in \mathbb{Z}$ satisfies $2^s \geq 2\,\Omega$. It turns out that ψ is a PF AB-wavelet, where $A = \{a_i = \left(\begin{smallmatrix} 2^i & 0 \\ 0 & 1 \end{smallmatrix}\right) : i \in \mathbb{Z}\}$ and $B = \{b_j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) : j \in \mathbb{Z}\}$. The proof of this fact is based on an application of a result from [7], where the characterization equations for a very general class of Parseval frames are obtained.

As we mentioned before, there are many choices for the functions ψ_1 and ψ_2 that satisfy the assumptions we have described above. For example, we can choose ψ_1 to be the Lemariè-Meyer wavelet (see [8, Sec. 1.4]) defined by $\hat{\psi}_1(\xi) = e^{i\pi\xi}\, b(\xi)$, where

$$b(\xi) = \begin{cases} \sin\big(\frac{\pi}{2}\,(3|\xi| - 1)\big), & \frac{1}{3} \leq |\xi| \leq \frac{2}{3}, \\ \sin\big(\frac{3\pi}{4}\,(\frac{4}{3} - |\xi|)\big), & \frac{2}{3} < |\xi| \leq \frac{4}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

In order to construct ψ_2, let ϕ be a compactly supported C^∞ bump function, with $\operatorname{supp}\phi \subset [-1,1]$ (examples can be found in [10, Sec. 3.3] or [9, Sec. 1.4]), and define ψ_2 by

$$\hat{\psi}_2(\xi) = \frac{\phi(\xi)}{\sqrt{\sum_{k\in\mathbb{Z}} |\phi(\xi+k)|^2}}\,.$$

It is clear that $\psi_2 \in C^\infty(\mathbb{R})$ and satisfies (3.1). It follows that $\hat{\psi}$, given by (3.2), is in $C^\infty(\mathbb{R}^2)$, and this implies that $|\psi(x)| \le K_N\,(1+|x|)^{-N}$, $K_N > 0$, for any $N \in \mathbb{N}$.

The following example shows how to construct MRA AB-wavelets for $L^2(\mathbb{R}^2)$ which are well-localized both in $\mathbb{R}^n$ and $\widehat{\mathbb{R}}^n$.

3.4. **Example 4.** Let $\psi_1 \in L^2(\mathbb{R})$ be a (one-dimensional) dyadic band-limited MRA wavelet with $\operatorname{supp}\hat{\psi}_1 \subset [-\Omega,\Omega]$, $\Omega > 0$, and ϕ_1 be its associated scaling function. Let m_0 and m_1 be the low pass and high pass filters, respectively, associated with ϕ_1 and ψ_1, that is, m_0, m_1 are the periodic functions satisfying the equations

$$\hat{\phi}_1(\omega_1) = m_0(\frac{\omega_1}{2})\,\hat{\phi}_1(\frac{\omega_1}{2}) \quad \text{and} \quad \hat{\psi}_1(\omega_1) = m_1(\frac{\omega_1}{2})\,\hat{\phi}_1(\frac{\omega_1}{2}).$$

Let $\psi_2 \in L^2(\mathbb{R})$ be defined by

$$\psi_2(x) = e^{i(N+1)\pi x}\,\Big(\frac{\sin \pi x}{\pi x}\Big)^{N+1},$$

where $N \in \mathbb{N}$. That is, $\hat{\psi}_2$ is a basic spline of order N (cf. [8]).

For $\omega = (\omega_1,\omega_2) \in \mathbb{R}^2$, $\omega_1 \ne 0$, let $\phi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\phi}(\omega) = \hat{\phi}_1(2^s\,\omega_1)\,\frac{\hat{\psi}_2(\frac{\omega_2}{\omega_1})}{\sqrt{\sum_{m\in\mathbb{Z}}\big|\hat{\psi}_2(\frac{\omega_2}{\omega_1}+m)\big|^2}},$$

where $s \in \mathbb{Z}$ satisfies $2^s \ge 4\,\Omega\,(\frac{N}{2}+1)$. This assumption on s ensures that

$$\operatorname{supp}\Big\{\hat{\phi}_1(2^s\,\omega_1)\,\hat{\psi}_2(\frac{\omega_2}{\omega_1})\Big\} \subset [-\frac{1}{4},\frac{1}{4}]^2.$$

Also, let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\omega) = \sum_{k=0}^{N+1} d_k^{(N)}\,m_1(2^{s-1}\omega_1)\,M_0(a^{-1}\omega)\,\hat{\phi}((b^T)^{-k}a^{-1}\,\omega),$$

where the matrices a and b are as in Section 3.3, $d_k^{(N)} = 2^{-N}\binom{N+1}{k}$, and $M_0(\omega)$ is the $\mathbb{Z}^2$-periodic function which, restricted to the fundamental region $[-\frac{1}{2},\frac{1}{2}]^2$, is given by

$$M_0(\omega) = \left(\frac{\sum_{m\in\mathbb{Z}}\big|\hat{\psi}_2(\frac{\omega_2}{\omega_1}+m)\big|^2}{\sum_{m\in\mathbb{Z}}\big|\hat{\psi}_2(2^{-1}\,\frac{\omega_2}{\omega_1}+m)\big|^2}\right)^{1/2}, \qquad \omega \in [-\frac{1}{2},\frac{1}{2}]^2.$$

It turns out that ψ is a PF MRA AB-wavelet for $L^2(\mathbb{R}^2)$. Indeed, the spaces $\{V_j : j \in \mathbb{Z}\}$, where $V_j = D_a^{-j}V_0$, $j \in \mathbb{Z}$, and $V_0 = \overline{\operatorname{span}\{D_b\,T_m\,\phi : b \in B, m \in \mathbb{Z}^2\}}$, form an AB-MRA. Observe, however, that this system is somewhat different from those described in Section 2, since the spaces V_0 and $W_0 = \overline{\operatorname{span}\{D_b\,T_m\,\psi : b \in B, m \in \mathbb{Z}^2\}}$ are *not* mutually orthogonal. Also, by construction, $\hat{\psi} \in C^N(\widehat{\mathbb{R}}^2)$, so $|\psi(x)| \le K_N\,(1+|x|)^{1-N}$, for some $K_N > 0$.

3.5. **Extensions for $n > 2$.** For $n > 2$, there are several generalizations of the examples described in the previous sections. The general idea is to write $\widehat{R}^n = F \oplus E$ and to define matrices a, b such that:

(i) $Ea \subseteq E$;

(ii) the induced action of a on $\widehat{R}^n / E$ is expanding;

(iii) b is the identity map on E and $b - I$ maps F to E.

This allows us to construct a space $V_0 = L^2(S)^\vee$ as we did in the 2-dimensional examples, for strip domains $S = c \times E$, where $c \subset F$ is contained in a neighborhood of the origin.

In particular, in order to extend Examples 1 and 2, let $n = k + \ell$, and $A = \{a^i : i \in \mathbb{Z}\}$, where $a = \left(\begin{smallmatrix} a_0 & a_1 \\ 0 & a_2 \end{smallmatrix}\right) \in GL_n(\mathbb{R})$, and $a_0 \in GL_k(\mathbb{R})$ is an expanding matrix. The matrices B can be chosen according to one of the following patterns.

(i) Let $k = 1$, $\ell = n - 1$ and $B = \{b_j = \left(\begin{smallmatrix} 1 & j \\ 0 & I_\ell \end{smallmatrix}\right) : j \in \mathbb{Z}^\ell\}$. Then $\psi = (\chi_R)^\vee$, where $R = \{(\xi_0, \dots, \xi_\ell) \in \mathbb{R}^n : \frac{1}{2|a_0|} \le |\xi_0| < \frac{1}{2}, 0 \le \xi_i/\xi_0 < 1, \text{ for } 1 \le i \le \ell\}$, is a PF MRA AB-wavelet.

(ii) Let $k \ge \ell > 0$ and

$$B = \left\{ b_{j_1,\dots,j_\ell} = \begin{pmatrix} I_{k-\ell} & 0 & 0 \\ 0 & I_\ell & \begin{pmatrix} j_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & j_\ell \end{pmatrix} \\ 0 & 0 & I_\ell \end{pmatrix} : j_i \in \mathbb{Z}, \text{ for each } 1 \le i \le \ell \right\}.$$

To illustrate this example, let $k = 3$, $\ell = 2$, and a_0 be a diagonal matrix with diagonal entries c_0, c_1, c_2 and $|c_i| > 1$, $i = 0, 1, 2$. Then $\psi = (\chi_R)^\vee$, where $R = \{(\xi_0, \dots, \xi_4) \in \mathbb{R}^n : \frac{1}{2|c_i|} \le |\xi_i| < \frac{1}{2}, i = 0, 1, 2, \text{ and } 0 \le \xi_{i+2}/\xi_i < 1, \text{ for } i = 1, 2\}$, is a PF MRA AB wavelet. The construction is similar for any $k \ge \ell$.

(iii) Let $k = 1$, $\ell = n - 1$ and

$$B = \left\{ b_{j_1,\dots,j_\ell} = \begin{pmatrix} 1 & j_1 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & j_\ell \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : j_i \in \mathbb{Z}, 1 \le i \le \ell \right\}.$$

Then $\psi = (\chi_R)^\vee$, where $R = \{(\xi_0, \dots, \xi_\ell) \in \mathbb{R}^n : \frac{1}{2|a_0|} \le |\xi_0| < \frac{1}{2}, 0 < \xi_i/\xi_{i-1} < 1, \text{ for } 1 \le i \le \ell\}$, is a PF MRA AB wavelet. Observe that, unlike the cases (i) and (ii), here the matrices B do not form a group.

In order to generalize Examples 3 and 4 to dimensions $n > 2$, we can proceed as follows. Again, let $n = k + \ell$ and $A = \{a^i : i \in \mathbb{Z}\}$, where

$$a = \begin{pmatrix} \begin{pmatrix} a_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & a_k \end{pmatrix} & 0 \\ 0 & I_\ell \end{pmatrix} \in GL_n(\mathbb{R})$$

and $|a_i| > 1$, for each $i = 1, \dots, k$. The matrices B can be chosen according to one of the following patterns.

(i) Let $k=1$, $\ell=n-1$ and $B=\{b_j=\left(\begin{smallmatrix}1 & j\\ 0 & I_\ell\end{smallmatrix}\right) : j=(j_1,\dots,j_\ell)\in\mathbb{Z}^\ell\}$. Let $\psi_1\in L^2(\mathbb{R})$ be a (one-dimensional) band-limited wavelet with respect to the dilation a_1, having $\operatorname{supp}\hat\psi_1\subset[-\Omega,\Omega]$, $\Omega>0$, and let $\psi_2,\dots,\psi_n\in L^2(\mathbb{R})$ be also band-limited functions with $\operatorname{supp}\hat\psi_i\subset[-1,1]$, $i=2,\dots,n$, and satisfying

$$\sum_{j\in\mathbb{Z}}|\hat\psi_i(\xi+j)|^2=1 \qquad \text{for a.e. } \xi\in\mathbb{R}, \text{ for any } i=2,\dots,n. \tag{3.3}$$

Now, for any $\omega=(\omega_1,\dots,\omega_n)\in\mathbb{R}^n$, $\omega_1\neq0$, define $\psi\in L^2(\mathbb{R}^n)$ by

$$\hat\psi(\omega)=\hat\psi_1(a_1^s\,\omega_1)\,\hat\psi_2\Big(\frac{\omega_2}{\omega_1}\Big)\times\cdots\times\hat\psi_n\Big(\frac{\omega_n}{\omega_1}\Big),$$

where $s\in\mathbb{Z}$ satisfies $a_1^s\geq 2\,\Omega$. Then one can show that ψ is a PF AB-wavelet.

(ii) Let $n=2k$ and

$$B=\left\{b_j=\begin{pmatrix} I_k & \begin{pmatrix} j_1 & \dots & 0\\ & \ddots & \\ 0 & \dots & j_k\end{pmatrix}\\ 0 & I_k\end{pmatrix} : j=(j_1,\dots,j_k)\in\mathbb{Z}^k\right\}.$$

Let $\psi_1,\dots,\psi_k\in L^2(\mathbb{R})$ be band-limited wavelets with respect to the dilations $a_1,\dots,a_k$, having $\operatorname{supp}\hat\psi_i\subset[-\Omega_i,\Omega_i]$, $\Omega_i>0$, and let $\psi_{k+1},\dots,\psi_n\in L^2(\mathbb{R})$ be also band-limited functions with $\operatorname{supp}\hat\psi_i\subset[-1,1]$, $i=k+1,\dots,n$, and satisfying (3.3) for any $i=k+1,\dots,n$. Now, for any $\omega=(\omega_1,\dots,\omega_n)\in\mathbb{R}^n$, $\omega_1,\dots,\omega_k\neq0$, define $\psi\in L^2(\mathbb{R}^n)$ by

$$\hat\psi(\omega)=\hat\psi_1(a_1^{s_1}\,\omega_1)\times\cdots\times\hat\psi_k(a_k^{s_k}\,\omega_k)\,\hat\psi_{k+1}\Big(\frac{\omega_{k+1}}{\omega_1}\Big)\times\cdots\times\hat\psi_n\Big(\frac{\omega_{2k}}{\omega_k}\Big),$$

where $s_k\in\mathbb{Z}$ satisfies $a_k^{s_k}\geq 2\,\Omega_k$. Then one can show that ψ is a PF AB-wavelet.

3.6. **Composite wavelet sets.** In this section, we consider AB-wavelets where $\psi\in L^2(\mathbb{R}^n)$ is given by $\hat\psi=\chi_S$, with $S\subset\widehat{\mathbb{R}}^n$. These systems are the analog of the so-called MSF wavelets in the classical wavelet theory.

As a specific example of such systems, consider, for simplicity, the case $n=2$. Let $B=\{b_j=\left(\begin{smallmatrix}1 & j\\ 0 & 1\end{smallmatrix}\right) : j\in\mathbb{Z}\}$, and $A=\{a^i : i\in\mathbb{Z}\}$, where $a=\left(\begin{smallmatrix}a_1 & 0\\ 0 & a_2\end{smallmatrix}\right)$, with $|a_1|>1$ and $a_2\neq0$. Given any $0<\delta\leq1$, let $S=\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \delta\leq|\xi_1|\leq\delta\,|a_1|, \text{ and } 0\leq\xi_2\leq|\xi_1|\}$, and define ψ by $\hat\psi=\chi_S$. A direct computation shows that

$$\Omega=\bigcup_{i,j\in\mathbb{Z}} S\,b^j\,a^i=\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi\neq0\}, \tag{3.4}$$

where the union is disjoint, and this implies that ψ is a PF AB-wavelet.

If, in addition, we take $a\in GL_n(\mathbb{Z})$, then, by choosing S appropriately, we can construct systems which are not only PF but even ON bases for $L^2(\mathbb{R}^n)$. For example, let $T(0,\alpha)$ be as in Example 1, and let $T(0,\alpha)+(\beta,\gamma)$ denote the triangle obtained by translating $T(0,\alpha)$ by $(\beta,\gamma)\in\mathbb{R}^2$. Let $a=\left(\begin{smallmatrix}2 & 0\\ 0 & 2\end{smallmatrix}\right)$, B as above, and ψ

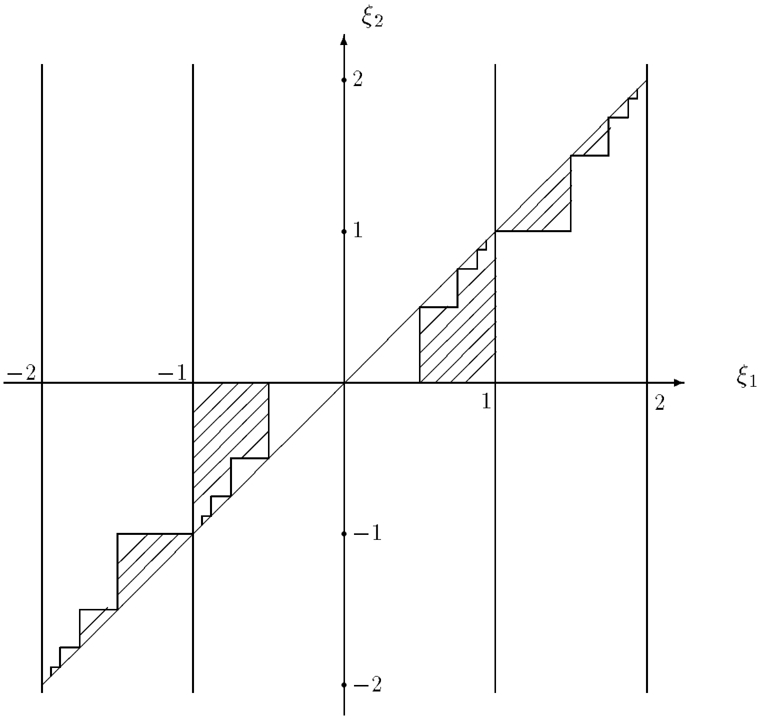

FIGURE 2. A singly generated ON AB-wavelet (Section 3.6).

be given by $\hat{\psi} = \chi_S$, where $S = S^+ \cup S^-$, $S^- = -S^+$ and S^+ is the subset of $\mathbb{R}^2$ equal to

$$\Big[T(0,1)\setminus\bigcup_{k=1}^{\infty}\{T(0,2^{-k})+(1-2^{-k+1})(1,1)\}\Big]\cup\Big[\bigcup_{k=1}^{\infty}\{T(0,2^{-k})+(2-2^{-k+1})(1,1)\}\Big].$$

This construction is illustrated in Figure 2. It is easy to see that S is a fundamental domain for the $\mathbb{Z}^2$-translations and equation (3.4) is satisfied. This shows that ψ is an ON AB-wavelet.

Observe that, since this orthonormal system is generated by a single function, it follows from Theorem 2.2 that ψ is **not** an ON MRA AB-wavelet.

For $a = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, one can construct an analogous unbounded set S. Again, ψ, where $\hat{\psi} = \chi_S$, is not an ON MRA AB-wavelet by Theorem 2.2. Higher-dimensional generalizations of these examples can also be obtained.

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