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RECENT PROGRESS ON THE BOUNDARY RIGIDITY PROBLEM

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Abstract. The boundary rigidity problem consists in determining a compact, Riemannian manifold with boundary, up to isometry, by knowing the boundary distance function between boundary points. In this paper we announce the result of our forthcoming article that one can solve this problem for generic simple metrics. Moreover we probe stability estimates for this problem.

1. Main results

Let $(M, \partial M, g)$ be a compact Riemannian manifold with boundary. Denote by ρ_g the distance function in the metric g. We consider the inverse problem of whether $\rho_g(x, y)$, known for all x, y on ∂M, determines the metric uniquely. It is clear that any isometry which is the identity at the boundary will give rise to the same distance functions on the boundary. Therefore, the natural question is whether this is the only obstruction to uniqueness. This is known in differential geometry as the boundary rigidity problem. The boundary distance function only takes into account the shortest paths, and it is easy to find counterexamples where ρ_g does not carry any information about certain open subset of M, so one needs to pose some restrictions on the metric. One such condition is simplicity of the metric.

Definition 1.1. We say that the Riemannian metric g is *simple* in M, if ∂M is strictly convex with respect to g, and for any $x \in M$, the exponential map $\exp_x : \exp_x^{-1}(M) \to M$ is a diffeomorphism.

Michel [13] conjectured that a *simple* metric g is uniquely determined, up to an action of a diffeomorphism fixing the boundary, by the boundary distance function $\rho_g(x, y)$ known for all x and y on ∂M.

This problem also arose in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves. It goes back to Herglotz [11] and Wiechert and Zoeppritz [30]. Although the emphasis has been on the case that the medium is isotropic, the anisotropic case has been of interest in geophysics since it has been found that the inner core of the Earth exhibits anisotropic behavior [6].

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Note that a simple metric g in M can be extended to a simple metric in some M_1 with $M \subset\subset M_1$. If we fix $x = x_0 \in M$ above, we also obtain that each simple manifold is diffeomorphic to a (strictly convex) domain $\Omega \subset \mathbf{R}^n$ with the Euclidean coordinates x in a neighborhood of Ω and a metric $g(x)$ there. For this reason, it is enough to prove our results for domains Ω in $\mathbf{R}^n$.

A closely related problem is to recover g from the *scattering relation* that maps initial points $x \in \partial M$ and directions ξ of maximal geodesics through M into outgoing points $y \in \partial M$ and directions η. Weaker geometric assumptions are needed to formulate this problem, and in the case of simple metrics, it is equivalent to the boundary rigidity problem. The scattering relation describes propagation of singularities of the corresponding wave equation through M, and is encoded in the hyperbolic *Dirichlet-to Neumann* map, and in the scattering operator in the case when M is embedded in $\mathbf{R}^n$.

Unique recovery of g (up to an action of a diffeomorphism) is known for simple metrics conformal to each other [8], [4], [14], [15], [16], [2], for flat metrics [10], for locally symmetric spaces of negative curvature [3]. In two dimensions it was known for simple metrics with negative curvature [7] and [17], and recently it was shown in [19] for simple metrics with no restrictions on the curvature. In [23], the authors proved a local result for metrics in a small neighborhood of the Euclidean one. This result was used in [12] to prove a semiglobal solvability result. Burago and Ivanov have shown recently that metrics close to the Euclidean metric are boundary rigid [5].

It is known [20], that a linearization of the boundary rigidity problem near a simple metric g is given by the following integral geometry problem: recover a symmetric tensor of order 2, which in any coordinates is given by $f = (f_{ij})$, by the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,\mathrm{d}t$$

known for all geodesics γ in M. It can be easily seen that $I_g dv = 0$ for any vector field v with $v|_{\partial M} = 0$, where dv denotes the symmetric differential

$$[dv]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \tag{1.1}$$

and $\nabla_k v$ denote the covariant derivatives of the vector field v. This is the linear version of the fact that ρ_g does not change on $(\partial M)^2 := \partial M \times \partial M$ under an action of a diffeomorphism as above. The natural formulation of the linearized problem is therefore that $I_g f = 0$ implies $f = dv$ with v vanishing on the boundary. We will refer to this property as *s-injectivity* of I_g. More precisely, we have.

Definition 1.2. We say that I_g is *s-injective* in M, if $I_g f = 0$ and $f \in L^2(M)$ imply $f = dv$ with some vector field $v \in H_0^1(M)$.

Any symmetric tensor $f \in L^2(M)$ admits an orthogonal decomposition $f = f^s + dv$ into a *solenoidal* and *potential* parts with $v \in H_0^1(M)$, and f^s divergence free, i.e., $\delta f^s = 0$, where δ is the adjoint operator to $-d$ given by $[\delta f]_i = g^{jk}\nabla_k f_{ij}$. Therefore, I_g is s-injective, if it is injective on the space of solenoidal tensors.

The inversion of I_g is a problem of independent interest in integral geometry, and our first two theorems are related to it. The s-injectivity of I_g was proved in [18] for metrics with negative curvature, in [20] for metrics with small curvature, and in [22] for Riemannian surfaces with no focal points. A conditional and non-sharp stability estimate for metrics with small curvature is also established in [20]. This estimate was used in [9] to get local uniqueness results for the boundary rigidity problem under the same condition. In [24], we proved stability estimates for s-injective metrics (see (1.2) below) and sharp estimates about the recovery of a 1-form $f = f_j dx^j$ and a function f from the associated $I_g f$. The stability estimates proven in [24] were used to prove local uniqueness for the boundary rigidity problem near any simple metric g with s-injective I_g.

Similarly to [28], we say that f is analytic in the set K (not necessarily open), if it is real analytic in some neighborhood of K. Our first main result is about s-injectivity at simple analytic metrics.

Theorem 1.3. *Let g be a simple, real analytic metric in M. Then I_g is s-injective.*

As shown in [24], the s-injectivity of I_g for analytic simple g implies a stability estimate for I_g. In the next theorem we show something more, namely that we have a stability estimate for g in a neighborhood of each analytic metric, which leads to stability estimates for generic metrics.

Let $M_1 \supset M$ be a compact manifold which is a neighborhood of M, and suppose g extends as a simple metric there. We always assume that our tensors are extended as zero outside M, which may create jumps at ∂M. Define the *normal operator* $N_g = I_g^* I_g$, where I_g^* denotes the operator adjoint to I_g with respect to an appropriate measure. We showed in [24] that N_g is a pseudodifferential operator in M_1 of order -1.

As in [24], we define the space $\tilde{H}^2(M_1)$ that in particular satisfies $H^2(M_1) \subset \tilde{H}^2(M_1) \subset H^1(M_1)$ (we refer to [24] for details). On the other hand, $f \in H^1(M)$ implies $N_g f \in \tilde{H}^2(M_1)$ despite the possible jump of f at ∂M.

Theorem 1.4. *There exists k_0 such that for each $k \geq k_0$, the set $\mathcal{G}^k(M)$ of simple $C^k(M)$ metrics in M for which I_g is s-injective is open and dense in the $C^k(M)$ topology. Moreover, for any $g \in \mathcal{G}^k$,*

$$\|f^s\|_{L^2(M)} \leq C\|N_g f\|_{\tilde{H}^2(M_1)}, \quad \forall f \in H^1(M), \tag{1.2}$$

with a constant $C > 0$ that can be chosen locally uniform in $\mathcal{G}^k$ in the $C^k(M)$ topology.

Of course, $\mathcal{G}^k$ includes all real analytic simple metrics in M, according to Theorem 1.3.

The analysis of I_g can also be carried out for symmetric tensors of any order; see e.g. [20] and [21]. Since we are motivated by the boundary rigidity problem, and to simplify the exposition, we study only tensors of order 2.

Theorem 1.4 and especially estimate (1.2) allow us to prove the following local generic uniqueness result for the non-linear boundary rigidity problem.

Theorem 1.5. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 1.4. There exists $k \geq k_0$ such that for any $g_0 \in \mathcal{G}^k$, there is $\varepsilon > 0$ such that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C^k(M)} \leq \varepsilon$, $m = 1, 2$, we have the following:*

$$\rho_{g_1} = \rho_{g_2} \text{ on } (\partial M)^2 \quad \text{implies } g_2 = \psi_* g_1 \tag{1.3}$$

with some $C^{k+1}(M)$-diffeomorphism $\psi : M \to M$ fixing the boundary.

Finally, we prove a conditional stability estimate of Hölder type. A similar estimate near the Euclidean metric was proven in [29] based on the approach in [23].

Theorem 1.6. *Let k_0 and $\mathcal{G}^k(M)$ be as in Theorem 1.4. Then for any $\mu < 1$, there exists $k \geq k_0$ such that for any $g_0 \in \mathcal{G}^k$, there is an $\varepsilon_0 > 0$ and $C > 0$ with the property that that for any two metrics g_1, g_2 with $\|g_m - g_0\|_{C(M)} \leq \varepsilon_0$, and $\|g_m\|_{C^k(M)} \leq A$, $m = 1, 2$, with some $A > 0$, we have the following stability estimate:*

$$\|g_2 - \psi_* g_1\|_{C^2(M)} \leq C(A)\|\rho_{g_1} - \rho_{g_2}\|^{\mu}_{C(\partial M\times\partial M)}$$

with some diffeomorphism $\psi : M \to M$ fixing the boundary.

Theorem 1.6 can be used to obtain stability near generic simple metrics for the inverse problem of recovering g from the hyperbolic *Dirichlet-to-Neumann map* Λ_g. It is known that g can be recovered uniquely from Λ_g, up to a diffeomorphism as above, see e.g. [1]. This result however relies on a unique continuation theorem by Tataru [27] and it is unlikely to provide Hölder type of stability estimate as above. By using the fact that ρ_g is related to the leading singularities in the kernel of Λ_g, we can prove a Hölder stability estimate under the assumptions above, relating g and Λ_g. We refer to [25] for details.

2. Sketch of the main ideas

As mentioned above, we can assume that M is an open bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary, and g is a simple Riemannian metric there.

2.1. **S-injectivity for analytic metrics, Theorem 1.3.** The proof of Theorem 1.3 is based on the following. For analytic metrics, the normal operator $N_g = I_g^* I_g$ is an analytic pseudodifferential operator with a non-trivial null space. We construct an analytic parametrix that allows us to reconstruct the solenoidal part of a tensor field from its geodesic X-ray transform, up to a term that is analytic near Ω. If $I_g f = 0$, we show that for some v vanishing on $\partial\Omega$, $\tilde f := f - dv$ must be flat at $\partial\Omega$ and analytic in $\bar\Omega$, hence $\tilde f = 0$. This is similar to the known argument that an analytic elliptic pseudodifferential operator resolves the analytic singularities, hence cannot have compactly supported functions in its kernel. In our case we have a non-trivial kernel, and complications due to the presence of a boundary, in particular loss of one derivative.

2.2. **The a priori linear stability estimate, Theorem 1.4.** The proof of the basic estimate (1.2) is based on the following ideas. For g of finite smoothness, one can still construct a parametrix Q_g of N_g as above that allows us to reconstruct f^s from $N_g f$ up to smoothing operator terms. This is done in a way similar to that in [24] in two steps: first we invert microlocally N_g in a neighborhood Ω_1 of Ω, and that gives us $f^s_{\Omega_1}$, i.e., the solenoidal projection of f but related to Ω_1. Next, we compare $f^s_{\Omega_1}$ and f^s and show that one can get the latter from the former by an operator that loses one derivative. This is the same construction as in 2.1 above but the metric is only C^k, $k \gg 1$.

After applying the parametrix Q_g, the equation for recovering f^s from $N_g f$ is reduced to solving the Fredholm equation

$$(\mathcal{S}_g + K_g)f = Q_g N_g f, \quad f \in \mathcal{S}_g L^2(\Omega), \tag{2.1}$$

where $\mathcal{S}_g$ is the projection to solenoidal tensors; similarly we denote by $\mathcal{P}_g$ the projection onto potential tensors. Here K_g is a compact operator on $\mathcal{S}_g L^2(\Omega)$. We can write this as an equation in the whole $L^2(\Omega)$ by adding $\mathcal{P}_g f$ to both sides above to get

$$(I + K_g)f = (Q_g N_g + \mathcal{P}_g)f. \tag{2.2}$$

Then the solenoidal projection of the solution of (2.2) solves (2.1). A finite rank modification of K_g above can guarantee that $I + K_g$ has a trivial kernel, and therefore is invertible, if and only if N_g is s-injective. The problem then reduces to that of invertibility of $I + K_g$. The operators above depend continuously on $g \in C^k$, $k \gg 1$. Since for g analytic, $I + K_g$ is invertible by Theorem 1.3, it would still be invertible in a neighborhood of any analytic g, and estimate (1.2) is true with a locally uniform constant. Analytic (simple) metrics are dense in the set of all simple metrics, and this completes the sketch of the proof of Theorem 1.4.

2.3. **Generic local and global boundary rigidity, Theorem 1.5.** We prove Theorem 1.5 by linearizing and using Theorem 1.4, and especially (1.2); see also [24]. This requires first to pass to special semigeodesic coordinates related to each metric in which $g_{in} = \delta_{in}$, $\forall i$. We denote the corresponding pull-backs by g_1, g_2 again. Then we show that if g_1 and g_2 have the same distance on the boundary, then $g_1 = g_2$ on the boundary with all derivatives. As a result, for $f := g_1 - g_2$ we get that $f \in C_0^l(\bar\Omega)$ with $l \gg 1$ if $k \gg 1$; and $f_{in} = 0$, $\forall i$. Then we linearize to get

$$\|N_{g_1} f\|_{L^\infty(\Omega_1)} \le C\|f\|_{C^1}^2,$$

where $\Omega_1 \supset \bar\Omega$ is as above. Combine this with (1.2) and interpolation estimates, to get $\forall \mu < 1$,

$$\|f^s\|_{L^2} \le C\|f\|_{L^2}^{1+\mu}.$$

One can show that tensors satisfying $f_{in} = 0$ also satisfy $\|f\|_{L^2} \le C\|f^s\|_{H^2}$, and using this, and interpolation again, we get

$$\|f\|_{L^2} \le C\|f\|_{L^2}^{1+\mu'}, \quad \mu' > 0.$$

This implies $f = 0$ for $\|f\| \ll 1$. Note that the condition $f \in C_0^l(\bar\Omega)$ is used to make sure that f, extended as zero in $\Omega_1 \setminus \Omega$, is in $H_0^l(\Omega)$, and then use this fact in the interpolation estimates.

2.4. **The stability estimate.** To prove Theorem 1.6, we basically follow the uniqueness proof sketched above by showing that each step is stable. The analysis is more delicate near pairs of points too close to each other. An important ingredient of the proof is stability at the boundary, which is also of independent interest:

Theorem 2.1. *Let g_0 and g_1 be two simple metrics in Ω, and $\Gamma \subset\subset \Gamma' \subset \partial\Omega$ be two sufficiently small open subsets of the boundary. Then for some diffeomorphism ψ fixing the boundary,*

$$\left\|\partial_{x^n}^k(\psi_* g_1 - g_0)\right\|_{C^m(\bar\Gamma)} \le C_{k,m}\left\|\rho_{g_1}^2 - \rho_{g_0}^2\right\|_{C^{m+2k+2}\left(\overline{\Gamma'\times\Gamma'}\right)},$$

where $C_{k,m}$ depends only on Ω and on an upper bound of g_0, g_1 in $C^{m+2k+5}(\bar\Omega)$.

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