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About some properties of intermediate point in certain mean-value formulas

Dumitru Acu, Petrică Dicu

Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

In this paper we study a property of the intermediate point (see [10]) from the quadrature formula of the trapeze, Simpson's formula and the mean-value formula of N. Ciorănescu.

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1. In the specialized literature there are a lot of mean-value formulas (the mean-value theorem for derivates, the mean-value theorems for Riemann integrals, the quadrature formulas, the cubature formulas, etc.). In general they contain one or more intermediate points. This points have different properties which result from the properties of the classes of functions, to which the mean-value theorems refer to.

In [10] B. Jacobson demonstrated that for the mean-value formula

$$\int_a^b f(t)dt = (b-a)f(c),$$

where f is a continuous function on the real closed interval $[a,b]$, we have:

$$\lim_{b\to a}\frac{c-a}{b-a}=\frac{1}{2}$$

In [13] E. C. Popa finds the following results:

1.1 Let $f,g:[a,b]\to\mathbb{R}$, two continuous functions with f derivable in a, g non-negative and $f'(a)g(a)\neq 0$. From the first mean-value theorem for Riemann integral we have: for any $x\in(a,b]$ there is $c_x\in[a,x]$ such that:

$$\int_a^x f(t)g(t)dt = f(c_x)\cdot\int_a^x g(t)dt$$

Then:

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

1.2 Let $f,g:[a,b]\to\mathbb{R}$, f continuous, g monotone and derivable with $f(a)g'(a)\neq 0$. From the second mean-value for Riemann integral, we have: for any $x\in(a,b]$ there is $c_x\in[a,x]$ such that:

$$\int_a^x f(t)g(t)dt = g(a)\int_a^{c_x} f(t)dt + g(x)\int_{c_x}^x f(t)dt.$$

Then:

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

1.3 Let $f,g:[a,b]\to\mathbb{R}$, two continuous functions on $[a,b]$, derivable on (a,b), derivable two times in a and $g'(x)\neq 0$ for any $x\in[a,b]$, $f''(a)g'(a)\neq$

$f'(a)g''(a)$. By using Cauchy's theorem, for any $x \in [a,b]$ we have $g(a) \neq g(x)$ and there is $c_x \in (a,x)$ so that:

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_x)}{g'(c_x)}.$$

Then

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

In [5] S. Anita gives the following result: Let $x_0 \in I$, I interval and $f : I \to \mathbb{R}$ a function derivable of $(n+2)$ times on I and $f^{(n+2)}(x_0) \neq 0$. According to Taylor's theorem we have:

$$f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \tag{1}$$
$$+ \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c_x).$$

If for any $x \in I$ $\{x_0\}$ we fix $c_x \neq x_0$ for which relation (1) holds, then:

$$\lim_{x \to x_0} \frac{c_x - x_0}{x - x_0} = \frac{1}{n+2}.$$

V. Radu demonstrated in [14] that if $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ $(n+1)$ times derivable on I and $a, x \in I$, then there is c_x between a and x such that:

$$\frac{f(x) - \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}}{g(x) - \displaystyle\sum_{k=0}^{n} \frac{g^{(k)}(a)(x-a)^k}{k!}} = \frac{f^{(n+1)}(c_x)}{g^{(n+1)}(c_x)}. \tag{2}$$

In [4] D. Acu demonstrated that, for the intermediate point in formula (2), we have:

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{n+2}.$$

In this paper we study the property shown above for the intermediate point from the quadrature formula of the trapeze, the quadrature formula of Simpson and the mean-value formula of N. Ciorănescu [6].

2. First, let us consider the quadrature formula of the trapeze. If $f:[a,b]\to\mathbb{R}$, $f\in C^2[a,b]$, then for any $x\in(a,b]$ there is $c_x\in(a,x)$ such that:

$$\int_a^x f(t)dt=\frac{1}{2}(x-a)\left[f(a)+f(x)\right]-\frac{(x-a)^3}{12}f''(c_x). \tag{3}$$

(see [7], [9], [11], [12]).

In the above condition we have the following theorem:

Theorem 1. *If $f\in C^3[a,b]$ and $f'''(a)\neq 0$, then for the intermediate point c_x that appears in formula (3), it follows:*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

Proof. Let $F,G:[a,b]\to\mathbb{R}$ defined as follows:

$$F(x)=\int_a^x f(t)dt-\frac{1}{2}(x-a)[f(a)+f(x)]+\frac{(x-a)^3}{12}f''(a),\quad G(x)=(x-a)^4.$$

Since F and G are two times derivable on $[a,b]$ and $G'(x)\neq 0$, $G''(x)\neq 0$ for any $x\in(a,b]$, we have:

$$F'(x)=\frac{1}{2}f'(x)-\frac{1}{2}f(a)-\frac{1}{2}(x-a)f'(x)+\frac{1}{4}(x-a)^2f''(a)$$

$$F''(x)=-\frac{1}{2}(x-a)f''(x)+\frac{1}{2}(x-a)f''(a)=-\frac{1}{2}(x-a)[f''(x)-f''(a)].$$

$$G'(x)=4(x-a)^3,\ G''(x)=12(x-a)^2.$$

By using succesive l'Hospital rule, we obtain:

$$\lim_{x\to a}\frac{F''(x)}{G''(x)}=\lim_{x\to a}\frac{-\frac{1}{2}(x-a)[f''(x)-f''(a)]}{12(x-a)^2}=-\frac{1}{24}f'''(a)$$

Hence:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{24}f'''(a). \tag{4}$$

On the other hand:

$$\frac{F(x)}{G(x)}=\frac{-\frac{(x-a)^3}{12}f''(c_x)+\frac{(x-a)^3}{12}f''(a)}{(x-a)^4}=-\frac{1}{12}\frac{f''(c_x)-f''(a)}{x-a}=$$

$$=-\frac{1}{2}\frac{f''(c_x)-f''(a)}{c_x-a}\cdot\frac{c_x-a}{x-a}.$$

Hence:

$$\lim_{x\to a}\frac{F(x)}{G(x)}=-\frac{1}{12}f'''(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a} \tag{5}$$

because $c_x\to a$ when $x\to a$. From relations (4) and (5) we obtain:

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}$$

which is in fact the conclusion of Theorem 1.

3. Now we will consider Simpson's formula.

If $f:[a,b]\to\mathbb{R}$, $f\in C^4[a,b]$, then any $x\in(a,b]$, there is $c_x\in(a,b)$ such that:

$$\int_a^x f(t)dt=\frac{1}{6}(x-a)\left[f(a)+4f\left(\frac{a+x}{2}\right)+f(x)\right]-\frac{(x-a)^5}{2880}f^{(IV)}(c_x). \tag{6}$$

(see [7], [9], [11], [12]).

Our main result is contained in the following:

Theorem 2. *If $f \in C^5[a,b]$ and $f^{(5)}(a) \neq 0$, then for the intermediate point c_x which appears in formula (6) we have:*

$$\lim_{x\to a} \frac{c_x - a}{x - a} = \frac{1}{2}$$

Proof. Let us consider $F, G : [a,b] \to \mathbb{R}$ defined by:

$$F(x) = \int_a^x f(t)dt - \frac{1}{6}(x-a)\left[f(a) + 4f\left(\frac{a+x}{2}\right) + f(x)\right] + \frac{(x-a)^5}{2880} f^{(IV)}(a),$$

$$G(x) = (x-a)^6.$$

Since F and G are fifth times derivable on $[a,b]$, $G^{(k)}(x) \neq 0$ for $k = \overline{1,5}$, $\forall x \in (a,b]$ we have:

$$F^{(5)}(x) = \frac{1}{6} f^{(IV)}(x) - \frac{5}{24} f^{(IV)}\left(\frac{a+x}{2}\right) + \frac{1}{24} f^{(IV)}(a) -$$

$$-\frac{1}{6}(x-a)\left[\frac{1}{8} f^{(5)}\left(\frac{a+x}{2}\right) + f^{(5)}(x)\right],$$

$$G^{(5)}(x) = 6!(x-a).$$

We have:

$$\lim_{x\to a} \frac{F^{(5)}(x)}{G^{(5)}(x)} = \lim_{x\to a} \left\{ \frac{\frac{1}{6}\left[f^{(IV)}(x) - f^{(IV)}(a)\right]}{6!(x-a)} - \frac{\frac{5}{24}\left[f^{(IV)}\left(\frac{a+x}{2}\right) - f^{(IV)}(a)\right]}{\frac{a+x}{2} - a} \cdot \right.$$

$$\left. \cdot \frac{\frac{a+x}{2} - a}{6!(x-a)} - \frac{1}{6 \cdot 6!}\left[\frac{1}{8} f^{(5)}\left(\frac{a+x}{2}\right) + f^{(5)}(x)\right] \right\} =$$

$$= \frac{1}{6!}\left(\frac{1}{6} - \frac{5}{48} - \frac{3}{16}\right) f^{(5)}(a) = -\frac{1}{5! \cdot 48} f^{(5)}(a).$$

Hence:

$$\lim_{x\to a}\frac{F(x)}{G(x)} = -\frac{1}{5!\cdot 48}f^{(5)}(a). \tag{7}$$

On the other hand:

$$\frac{F(x)}{G(x)} = \frac{-\dfrac{(x-a)^5}{2880}f^{(IV)}(c_x)+\dfrac{(x-a)^5}{2880}f^{(IV)}(a)}{(x-a)^6} =$$

$$= -\frac{1}{2880}\frac{f^{(IV)}(c_x)-f^{(IV)}(a)}{c_x-a}\cdot\frac{c_x-a}{x-a}.$$

By applying the limit $x\to a$, we have $c_x\to a$ and we obtain:

$$\lim_{x\to a}\frac{F(x)}{G(x)} = -\frac{1}{2880}f^{(5)}(a)\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{8}$$

From the relations (7) and (8), it follow

$$\lim_{x\to a}\frac{c_x-a}{x-a} = \frac{1}{2}$$

what is exactly the assertion of Theorem 2.

4. The last part of the paper is dedicated to the study of the intermediate's point property from the mean-value formula of N. Ciorănescu [6].

Being given: a function $f:[a,b]\to\mathbb{R}$, $f\in C^m[a,b]$, a sequence of orthogonal polynomials $(p_n)_{n\geq 0}$ (degree of p_n is equal n) on $[a,b]$, in respect to a weight function $w:[a,b]\to(0,\infty)$, N. Ciorănescu demonstrated that the following formula is valid:

$$\int_a^b f(x)p_m(x)w(x)dx = \frac{f^{(m)}(c_b)}{m!}\int_a^b x^m p_m(x)w(x)dx. \tag{9}$$

Theorem 3. *If $f\in C^{(m+1)}[a,b]$ and $f^{(m+1)}(a)\neq 0$, then the intermediate point of the mean-value formula (9) satisfies the relation:*

$$\lim_{b\to a}\frac{c_b-a}{b-a} = \frac{1}{m+2}$$

Proof. To obtain this result we use the so-called "method of parameters" introduced by D. D. Stancu 46 years ago in [15] and [16], in order to construct quadrature formulae of high degree of exactness and by using the Hildebrand's V - method we will obtain the result from the theorem.

For more informations about the use of this effective method it can be consulted [1], [2], [3], [16].

First we have two results.

Lemma 1. *In orthogonality conditions of the* $(p_m)(x)$ *on* $[a,b] \subseteq \mathbb{R}$ *associated with the non-negative weight function* w *on* $[a,b]$ *we construct the polynoms* $(V_k)_{k=\overline{0,m}}$ *with Hildebrand's V - method (see [1], [2], [3], [8]) as follows*

$$V_0(x) = w(x)p_m(x),$$

$$V_j(x) = \int_a^x V_{j-1}(t)dt, j = \overline{1,m}.$$

Then: $V_j(a) = 0, V_j(b) = 0$*, for any* $j = \overline{1,m}$.

Proof. We have $V_j(a) = 0$, for any $j = \overline{1,m}$.

$$V_1(b) = \int_a^b V_0(t)dt = \int_a^b w(t)p_m(t)dt = 0$$

(from the orthogonality of the polynoms $(p_m)_{m\geq 0}$). Using the integration by parts, we obtain:

$$V_2(b) = \int_a^b V_1(t)dt = tV_1(t)|_a^b - \int_a^b tV_1'(t)dt = \int_a^b tV_0(t)dt = -\int_a^b tp_m(t)w(t)dt = 0.$$

In the same way, it follows

$$V_3(b) = \int_a^b V_2(t)dt = tV_2(t)|_a^b - \int_a^b tV_1(t)dt =$$

$$= tV_2(t)|_a^b - \frac{t^2}{2}V_1(t)|_a^b + \frac{1}{2}\int_a^b t^2 V_0(t)dt = 0.$$

$$\vdots$$

$$V_m(b) = \int_a^b V_{m-1}(t)dt = tV_{m-1}(t)|_a^b - \int_a^b tV_{m-2}(t)dt =$$

$$= -\frac{t^2}{2}V_{m-2}(t)|_a^b + \frac{1}{6}\int_a^b t^3 V_{m-3}(t)dt = ... = \frac{(-1)^{m-1}}{(m-1)!}\int_a^b t^{m-1}V_0(t)dt = 0.$$

Hence $V_j(b) = 0$, for any $j = \overline{1,m}$.

Lemma 2. *We have the following equalities:*

i) $\int\limits_a^b f(x)p_m(x)w(x)dx = (-1)^m \int\limits_a^b f^{(m)}(x)V_m(x)dx.$

ii) $\int\limits_a^b x^m V_m^{(m)}(x)dx = (-1)^m m! \int\limits_a^b V_m(x)dx.$

Proof. Using the integration by parts and taking into account Lemma 1, we have:

i) $\int\limits_a^b f(x)p_m(x)w(x)dx = \int\limits_a^b f(x)V_0(x)dx = \int\limits_a^b f(x)V_1'(x)dx = f(x)V_1(x)|_a^b -$

$$-\int_a^b f'(x)V_1(x)dx = \int_a^b f''(x)V_2(x)dx = ... = (-1)^m \int_a^b f^{(m)}(x)V_m(x)dx.$$

ii) $\int\limits_a^b x^m V_m^{(m)}(x)dx = \int\limits_a^b x^m (V_m^{(m-1)})'(x)dx = x^m V_m^{(m-1)}(x)|_a^b -$

$$-m\int\limits_a^b x^{m-1}V_m^{(m-1)}(x)dx = ... = (-1)^m m!\int\limits_a^b V_m(x)dx.$$

We now come back to the proof of the Theorem 3. Using the Lemma 2 results in relations (9) we obtain successively

$$\int\limits_a^b f(x)p_m(x)w(x)dx = (-1)^m\int\limits_a^b f^{(m)}(x)V_m(x)dx =$$

$$= \frac{f^{(m)}(c_b)}{m!}\int\limits_a^b x^m p_m(x)w(x)dx = \frac{f^{(m)}(c_b)}{m!}\int\limits_a^b x^m V_m^{(m)}(x)dx =$$

$$= \frac{f^{(m)}(c_b)}{m!}(-1)^m m!\int\limits_a^b V_m(x)dx.$$

Therefore:

$$\int\limits_a^b f^{(m)}(x)V_m(x)dx = f^{(m)}(c_b)\int\limits_a^b V_m(x)dx. \tag{10}$$

Now, we consider the functions $F, G : [a,b] \to \mathbb{R}$ defined by:

$$F(b) = \int\limits_a^b f^{(m)}(x)V_m(x)dx - f^{(m)}(a)\int\limits_a^b V_m(x)dx, \quad G(b) = (b-a)^{m+2}.$$

We have:

$$F(a) = 0, \quad F'(b) = f^{(m)}(b)V_m(b) - f^{(m)}(a)V_m(b) = 0,$$

$$F''(b) = f^{(m+1)}(b)V_m(b) + f^{(m)}(b)V_m'(b) - f^{(m)}(a)V_m'(b) =$$

$$= f^{(m+1)}(b)V_m(b) + f^{(m)}(b)V_{m-1}(b) - f^{(m)}(a)V_{m-1}(b) = 0.$$

$$\vdots$$

$$F^{(k)}(b) = f^{(m+k-1)}(b)V_m(b) + C_{k-1}^1 f^{(m+k-2)}(b)V_{m-1}(b) + ...+$$

$$+f^{(m)}(b)V_{m-k+1}(b) - f^{(m)}(a)V_{m-k+1}(b) = 0.$$

$$\vdots$$

$$F^{(m)}(b) = f^{(2m-1)}(b)V_m(b) + C_{m-1}^1 f^{(2m-2)}(b)V_{m-1}(b) + ...+$$

$$+f^{(m)}(b)V_1(b) - f^{(m)}(a)V_1(b) = 0.$$

$$F^{(m+1)}(b) = f^{(2m)}(b)V_m(b) + C_m^1 f^{(2m-1)}(b)V_{m-1}(b) + ... + C_m^m f^{(m)}(b)V_0(b)-$$

$$-f^{(m)}(a)V_0(b) = [f^{(m)}(b) - f^{(m)}(a)]V_0(b).$$

Analog: $G'(b) = (m+2)(b-a)^{m+1}, ..., G^{(m+1)}(b) = (m+2)!(b-a)$.

Therefore, using the l'Hospital rule, we obtain

$$\lim_{b\to a} \frac{F^{(m+1)}(b)}{G^{(m+1)}(b)} = \lim_{b\to a} \frac{[f^{(m)}(b) - f^{(m)}(a)]}{(m+2)!(b-a)} V_0(b) = \frac{f^{(m+1)}(a)}{(m+2)!} V_0(a).$$

According to the l'Hospital rules it follows

$$\lim_{b\to a} \frac{F(x)}{G(x)} = \frac{f^{(m+1)}(a)}{(m+2)!} V_0(a) \tag{11}$$

It is obvious that we have:

$$\frac{F(b)}{G(b)} = \frac{\int_a^b f^{(m)}(x)V_m(x)dx - f^{(m)}(a)\int_a^b V_m(x)dx}{(b-a)^{m+2}} =$$

$$= \frac{f^{(m)}(c_b)\int_a^b V_m(x)dx - f^{(m)}(a)\int_a^b V_m(x)dx}{(b-a)^{m+2}} =$$

$$= \frac{\int\limits_a^b V_m(x)dx}{(b-a)^{m+1}} \cdot \frac{f^{(m)}(c_b) - f^{(m)}(a)}{c_b - a} \cdot \frac{c_b - a}{b-a}.$$

By passing to limit with $b \to a$, we have $c_b \to a$, we will obtain:

$$\lim_{b\to a} \frac{F(b)}{G(b)} = \frac{V_0(a)}{(m+1)!} f^{(m+1)}(a) \cdot \lim_{b\to a} \frac{c_b - a}{b-a}, \tag{12}$$

because:

$$\lim_{b\to a} \frac{f^{(m)}(c_b) - f^{(m)}(a)}{b-a} = f^{(m+1)}(a),$$

and:

$$\lim_{b\to a} \frac{\int\limits_a^b V_m(x)dx}{(b-a)^{m+1}} = \lim_{b\to a} \frac{V_m(b)}{(m+1)(b-a)^m} = \lim_{b\to a} \frac{V_{m-1}(b)}{m(m+1)(b-a)^{m-1}} = ... =$$

$$= \lim_{b\to a} \frac{V_1(b)}{(m+1)!(b-a)} = \lim_{b\to a} \frac{V_0(b)}{(m+1)!} = \frac{V_0(a)}{(m+1)!}.$$

From relations (11) and (12) we have:

$$\lim_{b\to a} \frac{c_b - a}{b-a} = \frac{1}{m+2}$$

what is exactly the assertion of Theorem 3.

Remark 1. *For $w(x) = 1$ and $m = 0$ form Theorem 3 we obtain B. Jacobson's result.*

References

[1] D. Acu, *V-optimal quadrature formulas of Gauss - Christoffel type*, Calcolo, vol. 34, No. 1 - 4, 1997, 125 - 133.

[2] D. Acu, *Extremal problems in the numerical integration of the functions*, Doctor thesis (Cluj - Napoca), 1980 (in Romanian).

[3] D. Acu, *On the D. D. Stancu method of parameters*, Studia Univ. "Babeş - Bolyai", Mathematica, vol. XLII, nr. 1, March, 1997, 1 - 7.

[4] D. Acu, *About the intermediate point in the mean-value theorems*, The annual session of scientifical comunications of the Faculty of Sciences in Sibiu, 17 - 18 May, 2001 (in Romanian).

[5] S. Anita, *Problem C: 275, Gazeta Matematică*, nr. 9, 1987 (in Romanian).

[6] N. Ciorănescu, *La généralisation de la premiére formule de la moyenne*, L'einsegment Math., Genéve, 37 (1938), 292 - 302.

[7] A. Ghizzetti and A. Ossicini, *Quadrature Formulae*, Birkhäuser Verlag Basel und Stuttgart, 1970.

[8] F. B. Hildebrand, *Introduction to numerical analysis*, New York, Mc Graw - Hill, 1956.

[9] D. V. Ionescu, *Numerical quadratures*, Bucharest, Technical Editure, 1957 (in Romanian).

[10] B. Jacobson, *On the mean-value theorem for integrals*, The American Mathematical Monthly, vol. 89, 1982, 300 - 301.

[11] A. Lupaş, C. Manole, *Numerical analysis chapters*, Publishing House of the "Lucian Blaga" University of Sibiu, Sibiu, 1994 (in Romanian).

[12] A. Lupaş, *Numerical Methods*, Constant Publishing House, Sibiu, 1994 (in Romanian).

[13] E. C. Popa, *An intermediate point property in some the mean - value theorems*, Astra Matematică, vol. 1, nr. 4, 1990, 3 - 7 (in Romanian).

[14] V. Radu, *Elementary mathematics lessons*, Spiridon Publishing House, 1996 (in Romanian).

[15] D. D. Stancu, *A method for construction of the quadrature formulas of a high degree exactness*, Comunic. Acad. R.P.R. (Bucharest), 4 (8), 1957, 349 - 358.

[16] D. D. Stancu, *Sur quelques formules générales de quadrature du type Gauss - Christoffel* , Matematica (Cluj), 1 (24), 1959, 16 - 182.

"Lucian Blaga" University of Sibiu
Department of Mathematics
Str. Dr. I. Raţiu, nr. 5 - 7
550012 - Sibiu, Romania
E-mail address: *depmath@ulbsibiu.ro*