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On the rate of convergence of certain approximation operators

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

In this paper we study some properties for a sequence $(P_n)_{n\geq 1}$ of linear and positive operators, defined by (2).

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1 Introduction

In [2] A. Lupaş proposed to study the following sequence of linear and positive operators

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^{[k,-1]}}{k! 2^k} f\left(\frac{k}{n}\right) \tag{1}$$

$x \geq 0, f : [0, \infty) \to \mathbb{R}$, where $(\alpha)^{[k,-1]} = \alpha(\alpha+1), ..., (\alpha+k-1),\ k \geq 1$, $\alpha^{[0,-1]} = 1$.

This L_n-operator has a form very similar with Szász - Mirakyan operators. O. Agratini investigate these operators in [1].

In the present paper we investigate the sequence of linear positive operators defined by

$$(P_n f)(x) = \left(\frac{1}{\beta+1}\right)^{\frac{nx}{\beta}} \sum_{k=0}^{\infty} \frac{(nx)^{[k,-\beta]}}{k!(\beta+1)^n} f\left(\frac{k}{n}\right), \tag{2}$$

$\beta > 0, x \geq 0, f : [0, \infty) \to \mathbb{R}$.

Remark 1.1. *For $\beta = 1$ we obtain the Lupaş operators (1). We observe that the Mirakyan operators*

$$(M_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^n}{k!} f\left(\frac{k}{n}\right) \tag{3}$$

may be obtained as a limiting case of our operators (2).

For $\beta \to 0$ we have $P_n \to M_n$.

2 Main result

Lema 2.1. *The operators P_n defined by (2) verify*

$$(P_n e_0)(x) = 1, (P_n e_1)(x) = x, (P_n e_2)(x) = x^2 + \frac{\beta+1}{n} x \tag{4}$$

Proof. Taking into account of (2) the relations (4) are valid.

Lema 2.2. *Let P_n be defined by (2). Then for $f \in C[0, \infty)$ one has $\lim_{n\to\infty} P_n f = f$ uniformly in any compact $K \subset [0, \infty)$.*

Proof. Our assertion follows directly from the Lemma 1 and the well-known theorem of Bohman - Korovkin.

Now we estimate the rate of convergence for smooth function.

Theorem 2.1. *Let P_n be defined by 2. Then for $f \in C^1[0,\infty)$ are found*

$$(5) \quad |(P_n f)(x) - f(x)| \leq \sqrt{x(\beta+1)}[1+\sqrt{x(\beta+1)}]n^{-\frac{1}{2}} w\left(f; \frac{1}{\sqrt{n}}\right)$$

where $w(f,\delta) = \sup\limits_{0<h\leq\delta} \sup\limits_{x\geq 0}[f(x+h) - f(x)]$ is the usual first modulus of continuity.

Proof. We consider

$$f\left(\frac{k}{n}\right) - f(x) = \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)(f'(\xi) - f'(x))$$

where $\xi(x,n,k) = \xi$ is an interior point of the interval determined by x and $\frac{k}{n}$.

We give

$$(P_n f)(x) - f(x) = f'(x) \sum_{k=0}^{\infty} p_{n,k}^{\beta}(x)\left(\frac{k}{n} - x\right) +$$

$$+ \sum_{k=0}^{\infty} p_{n,k}^{\beta}(x)\left(\frac{k}{n} - x\right)(f'(\xi) - f'(x))$$

where

$$p_{n,k}^{\beta}(x) = \left(\frac{1}{\beta+1}\right)^{\frac{nx}{\beta}} \frac{(nx)^{[k,-\beta]}}{k!(\beta+1)^k}.$$

We observe that

$$f'(x) \sum_{k=0}^{\infty} p_{n,k}^{\beta}(x)\left(\frac{k}{n} - x\right) = 0$$

and we have

$$|(P_nf)(x)-f(x)|\le\sum_{k=0}^{\infty}p_{n,k}^{\beta}(x)|\frac{k}{n}-x||f'(\xi)-f'(x)|$$

It is known that

$$|f'(\xi)-f'(x)|\le w(f';|\xi-x|)\le\left(1+\delta^{-1}\left|\frac{k}{n}-x\right|\right)w(f';\delta).$$

Hence

$$|(P_nf)(x)-f(x)|\le$$

$$\le\left[\left(\sum_{k=0}^{\infty}p_{n,k}^{\beta}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}+\delta^{-1}\sum_{k=0}^{\infty}p_{n,k}^{\beta}(x)\left(\frac{k}{n}-x\right)^2\right]w(f';\delta),$$

$$|(P_nf)(x)-f(x)|\le\left[\sqrt{\frac{x(\beta+1)}{n}}+\delta^{-1}\frac{x(\beta+1)}{n}\right]w(f';\delta).$$

By choosing $\delta=\frac{1}{\sqrt{n}}$ we obtain (5).

For $\beta=1$ we have

$$|(L_nf)(x)-f(x)|\le\sqrt{2x}(1+\sqrt{2x})n^{-\frac{1}{2}}w\left(f;\frac{1}{\sqrt{n}}\right)$$

(Lupaş operators) and for $\beta\to 0$ we obtain for Mirakyan operators

$$|(M_nf)(x)-f(x)|\le\sqrt{x}(1+\sqrt{x})n^{-\frac{1}{2}}w\left(f;\frac{1}{\sqrt{n}}\right).$$

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