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On the efficiency of some optimal quadrature formulas attached to some given quadrature formulas

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

In this paper one studies some quadrature formulas from the efficiency point of view, in the class of optimal quadrature formulas attached to some given quadratures.

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1 Introduction

In this paper one consider two types of optimal quadrature formulas families (with respect to the error) attached to some given quadrature formulas, obtained in [1], for the class $W^1_{L_2}(M;0,1)$. For each family, one shows that the optimal quadrature formula with the degree of exactness 1 has the highest efficiency.

First we present the necessary concepts.

2 Preliminaries

Let $X = L[a,b], X_0 \subseteq X$ and $S : X_0 \to \mathbb{R}$ the integration operator defined by

$$S(f) = \int_a^b f(x)dx$$

One considers a quadrature formula of the following form

$$S(f) = Q_n(f) + R_n(f), \tag{1}$$

where

$$Q_n(f) = \sum_{k=0}^{n} A_k f(x_k),$$

and $R_n(f)$ is the remainder term.

We suppose that the information operator $\mathcal{I} : X_0 \to \mathbb{R}^{n+1}$ has the form $\mathcal{I}(f) = (f(x_0), ..., f(x_n))$, $x_k \in [a,b]$, $k = \overline{0,n}$, with $x_i \neq x_k$ for $i \neq k$. $\mathcal{I}(f)$ is called the information of f. Also we suppose that the set of primitive operations is represented by $\mathcal{R} = \{+, -, *, /\}$.

We denote by α the algorithm that computes the term $Q_n(f)$, $\alpha : \mathcal{I}(X_0) \to \mathbb{R}$ and by $\mathcal{A}(S,\mathcal{I})$ the set of all such algorithms that solve the problem (X_0, S) with the information $\mathcal{I}$.

In order to get an ϵ-approximation of the solution of an integration problem, the information operator $\mathcal{I}$ must be ϵ-admissible and $\mathcal{R}$-admissible. $\mathcal{I}$ is ϵ-admissible if the information radius $r(S,\mathcal{I}) < \epsilon$, where $r(S,\mathcal{I}) = \sup\limits_{f \in X_0} \text{rad}\,(U(f))$, with $U(f) = \{S(\widetilde{f}) | \mathcal{I}(\widetilde{f}) = \mathcal{I}(f)\}$ the set of all solutions of functions with the same information. I is an admissible information operator with respect to $\mathcal{R}$ if $I(f)$ can be computed for all $f \in X_0$, with a finite number of operations from $\mathcal{R}$ (taking into account that some of the operations can be applied several times). Also the algorithm α must be ϵ-admissible and $\mathcal{R}$-admissible. The algorithm α is ϵ-admissible if the error $e(S,\mathcal{I},\alpha) \leq \epsilon$, where $e(S,\mathcal{I},\alpha) = \sup_{f\in X_0} |R_n(f)|$. The algorithm α is called $\mathcal{R}$-admissible if $\alpha(\mathcal{I}(f))$ can be computed for all $f \in X_0$, with a finite number of operations from $\mathcal{R}$, and some of them may be repeated.

Suppose that the information operator $\mathcal{I}$ is ϵ-admissible and $\mathcal{R}$-admissible. One denotes by $\mathcal{A}(S,\mathcal{I},\epsilon)$ the set of all algorithms $\mathcal{L} \in \mathcal{A}(S,\mathcal{I})$ which are ϵ-admissible and $\mathcal{R}$-admissible. Let $r_1, ..., r_m \in \mathcal{R}$ be the necessary operations to compute $I(f), f \in X_0$. The value

$$CPE(\mathcal{I}(f)) = \sum_{i=1}^{m} p_i CP(r_i),$$

where p_i is the performing number of the operation r_i and $CP(r_i)$ is the complexity of the operation r_i, is called the **complexity of the information** $\mathcal{I}(f)$. The value

$$CPE(\mathcal{I}) = \sup_{f\in X_0} CPE(\mathcal{I}(f))$$

is called the **information complexity**.

Also, let $\rho_1, ..., \rho_s \in \mathcal{R}$ be the necessary operations to compute $\alpha(\mathcal{I}(f))$. The value

$$CPC(\alpha(\mathcal{I}(f))) = \sum_{j=1}^{s} q_j CP(\rho_j),$$

where q_j is the performing number of the operation ρ_j and $CP(\rho_j)$ is the complexity of ρ_j, is called the **combinatorial complexity of the algorithm** α **for the function** $f \in X_0$. The value

$$CPC(\alpha) = \sup_{f\in X_0} CPC(\alpha(\mathcal{I}(f)))$$

is called the **combinatorial complexity of the algorithm** α.

Finally, the value $CPA(S,\mathcal{I},\alpha)$ (briefly $CPA(\alpha)$), defined by

$$CPA(\alpha) = CPE(\mathcal{I}) + CPC(\alpha),$$

is called the **analytic complexity of the algorithm** α for the integration problem (X_0, S) with the information $\mathcal{I}$, or **the analytic complexity** of the quadrature formula (1).

The number $p, p = p(\alpha)$, with the property that

$$\lim_{h\to 0} \frac{e(S,\mathcal{I},\alpha)}{h^p} = k, k \neq 0,$$

where k is a constant, is called **the order of approximation of the algorithm** α. The value

$$E(S,\mathcal{I},\alpha) = \frac{\log_2 p(\alpha)}{CPA(\alpha)} \tag{2}$$

is called **the efficiency of the algorithm** α, or the **efficiency of the quadrature formula (1)**.

Both the analytic complexity and the efficiency represent criteria to compare the quadrature formulas.

3 The efficiency of some optimal quadrature formulas attached to some given quadratures

Let $X = L[0,1], X_0 \subset X$, the integral operator $S(f) = \int\limits_0^1 f(x)dx$ and the quadrature formula

$$\int\limits_0^1 f(x)dx = \sum_{k=0}^{n-1} A_k f(x_k) + R_n(f), \tag{3}$$

with the exact evaluation of the remainder term

$$R_n(L[0,1], A_k, x_k) = \sup_{f \in L[0,1]} |R_n(f)|.$$

The following formula is called an optimal quadrature formula (with respect to the error) attached to the quadrature formula (3) for the class $L[0,1]$:

$$\int_0^1 f(x)dx = \sum_{k=0}^{n-1} A_k f(x_k) + \sum_{i=0}^{m-1} B_i f(y_i) + R_m(f), \tag{4}$$

where

$$\sup_{f \in L[0,1]} |R_m(f)| \text{ is minimum.}$$

We denote by α the algorithm that computes the term

$$\sum_{k=0}^{n-1} A_k f(x_k) + \sum_{i=0}^{m-1} B_i f(y_i).$$

As we are going to deal with some quadrature formulas for a given function $f \in X_0$, we compute the local analytic complexity

$$CPA(\alpha(\mathcal{I}(f))) = CPE(\mathcal{I}(f)) + CPC(\alpha(\mathcal{I}(f))) \tag{5}$$

instead of

$$CPA(\alpha) = \sup_{f \in X_0} CPA(\alpha(\mathcal{I}(f))).$$

We suppose that in order to obtain the value $CPE(\mathcal{I}(f))$ we have the same computational complexity of the values $f(x_k)$, for every $k = \overline{0, n-1}$, denoted by $CP(f)$, i.e.

$$CP(f(x_0)) = CP(f(x_1)) = ... = CP(f(x_{n-1})) = CP(f).$$

Also, we suppose that $CP(-) = CP(+)$.

We use the following result [2]:

If the quadrature formula (1) has the degree of exactness r, then its order of approximation is given by $p = r + 2$.

We shall consider two particular cases.

3.1. Let $X_0 = W^1_{L_2}(M;0,1) = \{f : [0,1] \to \mathbb{R}$, absolute continuous, $\left(\int\limits_0^1 |f'(x)|^2\right)^{\frac{1}{2}} \leq M\}$, and $W^1_{oL_2}(M;0,1) = \{f \in W^1_{L_2}(M;0,1),\ f(0) = 0\}$. We suppose that (3) is the optimal quadrature formula for the class $W^1_{oL_2}(M;0,1)$. D. Acu [1] obtained, for this quadrature formula, the optimal attached quadrature formula of the form (4), for the class $W^1_{L_2}(M;0,1)$, i.e.

$$(6)\ \int\limits_0^1 f(x)dx = \frac{2}{2m+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right) + \frac{a}{2}f(0) + \frac{1}{2}\left(\frac{2}{2n+1} - a\right) f(a) +$$

$$+R_n(f,a),$$

with the optimal estimation for the remainder term:

$$(7)\qquad R_n(W^1_{L_2}(M;0,1);a) =$$

$$= M\sqrt{\frac{1}{3(2n+1)^2} - \left(\frac{2}{2n+1} - a\right)\left(\frac{1}{2n+1} - \frac{a}{2}\right)\frac{a}{2}},$$

where a is a given constant in the interval $\left(0, \frac{2}{2n+1}\right]$.

For $a = \frac{1}{2n+1}$, from (6) and (7) one obtains [1] the optimal quadrature formula (M. Levin):

$$(8)\qquad \int\limits_0^1 f(x)dx = \frac{2}{2n+1}\sum_{k=0}^{n-1} f\left(\frac{2k+2}{2n+1}\right) +$$

$$+\frac{1}{2(2n+1)}\left[f(0) + f\left(\frac{1}{2n+1}\right)\right] + R_n\left(f, \frac{1}{2n+1}\right),$$

with

$$R_n\left(W^1_{L_2}(M;0,1);\frac{1}{2n+1}\right)=\frac{M}{(2n+1)\sqrt{3}}\sqrt{1-\frac{3}{4}\cdot\frac{1}{2n+1}}.$$

The quadrature formula (8) has the degree of exactness 1.

The quadrature formula (6) for which the estimation (7) is minimal is obtained for $a=\frac{2}{3}\cdot\frac{1}{2n+1}$, i.e.

$$(9)\qquad \int_0^1 f(x)dx=\frac{2}{2n+1}\sum_{k=0}^{n-1}f\left(\frac{2k+2}{2n+1}\right)+$$

$$+\frac{1}{3(2n+1)}\left[f(0)+2f\left(\frac{2}{3}\cdot\frac{1}{2n+1}\right)\right]+R_n\left(f,\frac{2}{3}\cdot\frac{1}{2n+1}\right),$$

with

$$R_n\left(W^1_{L_2}(M;0,1);\frac{2}{3}\cdot\frac{1}{2n+1}\right)=\frac{M}{(2n+1)\sqrt{3}}\sqrt{1-\frac{8}{9}\cdot\frac{1}{2n+1}}$$

We denote by α,α_1, respectively $\overline{\alpha}$ the algorithm which approximates $\int_0^1 f(x)dx$ according to (6), (8), respectively (9).

By (5) we obtain:

$$CPA(\alpha(\mathcal{I}(f)))=(n+2)CP(f)+(n+3)CP(+)+(n+4)CP(*)+2CP(/),$$

$$CPA(\alpha_1(\mathcal{I}(f)))=(n+2)CP(f)+(n+2)CP(+)+(n+3)CP(*)+2CP(/),$$

$$CPA(\overline{\alpha}(\mathcal{I}(f)))=(n+2)CP(f)+(n+2)CP(+)+(n+4)CP(*)+2CP(/).$$

Finally, by (2) we have

$$E(\alpha(\mathcal{I}(f)))=\frac{1}{CPA(\alpha(\mathcal{I}(f)))},$$

$$E(\alpha_1(\mathcal{I}(f)))=\frac{\log_2 3}{CPA(\alpha_1(\mathcal{I}(f)))},$$

$$E(\overline{\alpha}(\mathcal{I}(f))) = \frac{1}{CPA(\overline{\alpha}(\mathcal{I}(f)))}.$$

One concludes that:

Proposition 3.1. $E(\alpha(\mathcal{I}(f))) < E(\overline{\alpha}(\mathcal{I}(f))) < E(\alpha_1(\mathcal{I}(f)))$.

3.2. We suppose that (3) is the composite trapezoidal quadrature formula. D. Acu [1] obtained, in this case, the optimal attached quadrature formula, for the class $W^1_{L2}(M;0,1)$, i.e.

$$(10)\int_0^1 f(x)dx = \frac{1}{n}\left[\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) + \frac{1}{2}f(1)\right] + \frac{a}{2}f(0) + \frac{1}{2}\left(\frac{1}{n} - a\right)f(a) + $$
$$+R_n(f,a),$$

with the optimal estimation for the remainder term

$$(11) \qquad R_n(W^1_{L2}(M;0,1);a) = \frac{M}{2n\sqrt{3}}\sqrt{1-3(1-na)^2a},$$

where a is a fixed constant in the interval $\left(0, \frac{1}{n}\right]$.

For $a = \frac{1}{n}$, from (10) and (11) one obtains the optimal composite trapezoidal quadrature formula

$$(12) \qquad \int_0^1 f(x)dx = \frac{1}{n}\left[\frac{f(0)+f(1)}{2} + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\right] + R_n\left(f, \frac{1}{n}\right),$$

with the optimal estimation for the remainder term

$$R_n\left(W^1_{L_2}(M;0,1); \frac{1}{n}\right) = \frac{M}{2n\sqrt{3}}.$$

The quadrature formula (12) has the degree of exactness 1.

The best from the quadrature formula (10) is obtained for $a = \frac{1}{3n}$, i.e.

$$(13) \qquad \int_0^1 f(x)dx = \frac{1}{n}\left[\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) + \frac{1}{2}f(1) + \frac{1}{6}f(0) + \frac{1}{3}f\left(\frac{1}{3n}\right)\right] +$$

$$+R_n\left(f, \frac{1}{3n}\right),$$

with

$$R_n\left(W^1_{L_2}(M;0,1); \frac{1}{3n}\right) = \frac{M}{2n\sqrt{3}}\sqrt{1-\frac{4}{9n}}.$$

We denote by β, β_1, respectively $\overline{\beta}$ the algorithm which approximates $\int\limits_0^1 f(x)dx$ according to (10), (12), respectively (13).

By (5), from straightforward computation, we obtain

$$CPA(\beta(\mathcal{I}(f))) = (n+2)CP(f) + (n+2)CP(+) + (n+1)CP(*) + 4CP(/),$$

$$CPA(\beta_1(\mathcal{I}(f))) = (n+1)CP(f) + (2n-2)CP(+) + CP(*) + 2CP(/),$$

$$CPA(\overline{\beta}(\mathcal{I}(f))) = (n+2)CP(f) + (n+2)CP(+) + (n-2)CP(*) + 5CP(/).$$

For efficiencies, we have

$$E(\beta(\mathcal{I}(f))) = \frac{1}{CPA(\beta(\mathcal{I}(f)))},$$

$$E(\beta_1(\mathcal{I}(f))) = \frac{\log_2 3}{CPA(\beta_1(\mathcal{I}(f)))},$$

$$E(\overline{\beta}(\mathcal{I}(f))) = \frac{1}{CPA(\overline{\beta}(\mathcal{I}(f)))}.$$

So, we deduce that:

Proposition 3.2. $E(\beta(\mathcal{I}(f))) < E(\overline{\beta}(\mathcal{I}(f))) < E(\beta_1(\mathcal{I}(f)))$.

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