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On a particular second-order nonlinear differential subordination I

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Abstract

We find conditions on the complex-valued functions A, B, C, D in the unit disc U such that the differential inequality

$$|A(z)z^2p'(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In [1] chapter IV, the authors have analyzed a second-order linear differential subordination

$$(1) \qquad A(z)z^2p'(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where A, B, C, D and h are complex-valued functions in the unit disc, where $p \in \mathcal{H}[0, n]$. A more general version of (1) is given by:

$$A(z)z^2p'(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega,$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions A, B, C, D in the unit disc U and the positive numbers M and N such that

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$(2) \qquad |A(z)z^2p'(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M,$$

where $p \in \mathcal{H}[0, n]$.

We find conditions on complex-valued functions A, B, C, D and the positive numbers M and N such that (2) implies

$$|p(z)| < N,$$

where $p \in \mathcal{H}[0, n]$.

In order to prove the new results we shall use the following lemma:

Lemma A. [1, p. 34] *Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and* $M > 0$, $N > 0$, n *positive integer, satisfy*

$$|\psi(Ne^{i\theta}, Ke^{i\theta}, L; z)| \geq M \tag{3}$$

whenever

$$\text{Re } [Le^{-i\theta}] \geq (n-1)K, \quad K \geq nN,$$

$z \in U$ *and* $\theta \in \mathbb{R}$.

If $p \in \mathcal{H}[0, n]$ *and*

$$|\psi(p(z), zp'(z), z^2p'(z); z)| < M$$

then $|p(z)| < N$.

2 Main results

Theorem. *Let* $M > 0$, $N > 0$, *and let* n *be a positive integer. Suppose that the functions* $A, B, C, D : U \to \mathbb{C}$ *satisfy* $A(z) \neq 0$,

(i) $\text{Re } \dfrac{B(z)}{A(z)} \geq -n$

(ii) $\mathrm{Re}\,\dfrac{B(z)+D(z)}{A(z)} \geq \dfrac{M+N^2|C(z)|}{N|A(z)|}$.

If $p \in \mathcal{H}[0,n]$ *and*

$$|A(z)z^2p'(z)+B(z)zp'(z)+C(z)p^2(z)+D(z)p(z)|<M$$

then

$$|p(z)|<N.$$

Proof. Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ be defined by

$$\psi(p(z),zp'(z),z^2p'(z);z)=A(z)z^2p'(z)+B(z)zp'(z)+ \tag{4}$$

$$+C(z)p^2(z)+D(z)p(z).$$

From (2) we have

$$|\psi(p(z),zp'(z),z^2p'(z);z)|<M,\ \text{for } z\in U. \tag{5}$$

Using (ii) in (4) we have

$$|\psi(Ne^{i\theta},Ke^{i\theta},L;z)|=|A(z)L+B(z)+Ke^{i\theta}+C(z)N^2e^{2i\theta}+D(z)Ne^{i\theta}|=$$

$$=|A(z)Le^{-i\theta}+B(z)K+C(z)N^2e^{i\theta}+D(z)N|=$$

$$=|A(z)|\left|Le^{-i\theta}+\frac{B(z)}{A(z)}K+\frac{C(z)}{A(z)}N^2e^{i\theta}+\frac{D(z)}{A(z)}N\right|\geq$$

$$\geq|A(z)|\left[\left|Le^{-i\theta}+K\frac{B(z)}{A(z)}+N\frac{D(z)}{A(z)}\right|-N^2\left|\frac{C(z)}{A(z)}\right|\right]\geq$$

$$\geq|A(z)|\left[\mathrm{Re}\ Le^{-i\theta}+K\mathrm{Re}\ \frac{B(z)}{A(z)}+N\mathrm{Re}\ \frac{D(z)}{A(z)}-N^2\left|\frac{C(z)}{A(z)}\right|\right]\geq$$

$$\geq |A(z)|\left[(n-1)K + K\text{Re}\,\frac{B(z)}{A(z)} + N\text{Re}\,\frac{D(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right|\right] \geq$$

$$\geq |A(z)|\left[n(n-1)N + nN\text{Re}\,\frac{B(z)}{A(z)} + N\text{Re}\,\frac{D(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right|\right] \geq$$

$$\geq |A(z)|\left[nN\text{Re}\,\frac{B(z)}{A(z)} + N\text{Re}\,\frac{D(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right|\right] \geq$$

$$\geq |A(z)|\left[N\text{Re}\,\frac{B(z)}{A(z)} + N\text{Re}\,\frac{D(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right|\right] \geq$$

$$\geq |A(z)|\left[N\text{Re}\,\frac{B(z)+D(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right|\right] \geq M.$$

Hence condition (3) holds and by Lemma A we deduce that (5) implies $|p(z)| < N$.

Instead of prescribing the constant N in Theorem, in some cases we can use (ii) to determine an appropriate $N = N(M, n, A, B, C, D)$ so that (2) implies $|p(z)| < N$. This can be accomplished by solving (ii) for N by taking the supremum of the resulting function over U.

Condition (ii) is equivalent to:

$$N^2|C(z)| - N|A(z)|\text{Re}\,\frac{B(z)+D(z)}{A(z)} + M \leq 0. \tag{6}$$

If we suppose $C(z) \neq 0$, then the inequality (6) holds if

$$|A(z)|\text{Re}\,\frac{B(z)+D(z)}{A(z)} \geq 2\sqrt{|C(z)|M} \tag{7}$$

If (7) holds, the roots of the trinomial in (6) are

$$N_{1,2} = \frac{|A(z)|\text{Re}\,\frac{B(z)+D(z)}{A(z)} \pm \sqrt{\left[|A(z)|\text{Re}\,\frac{B(z)+D(z)}{A(z)}\right]^2 - 4M|C(z)|}}{2|C(z)|}.$$

We let

$$N=\frac{2M}{|A(z)|\mathrm{Re}\,\dfrac{B(z)+D(z)}{A(z)}+\sqrt{\left[|A(z)|\mathrm{Re}\,\dfrac{B(z)+D(z)}{A(z)}\right]^2-4M|C(z)|}}.$$

If this supremum is finite, the Theorem can be rewritten as follows:

Corollary 1. *Let $M>0$ and let n be a positive integer. Suppose that $p\in\mathcal{H}[0,n]$ and that the functions $A,B,C,D:U\to\mathbb{C}$, with $A(z)\neq 0$, satisfy*

(i) $\mathrm{Re}\,\dfrac{B(z)}{A(z)}\geq -n$

(ii) $N=\sup\limits_{|z|<1}\dfrac{2M}{|A(z)|\mathrm{Re}\,\frac{B(z)+D(z)}{A(z)}+\sqrt{\left[|A(z)|\mathrm{Re}\,\frac{B(z)+D(z)}{A(z)}\right]^2-4M|C(z)|}}<\infty$

then

$$|A(z)z^2p'(z)+B(z)zp'(z)+C(z)p^2(z)+D(z)p(z)|<M$$

implies

$$|p(z)|<N.$$

Let $n=1$, $A(z)=3$, $B(z)=2+4z$, $C(z)=2$, $D(z)=10-4z$, $M=8$, $N=\dfrac{4}{3+\sqrt{5}}$.

In this case from Corollary 1 we deduce:

Example 1. If $p\in\mathcal{H}[0,1]$, then

$$|3z^2p'(z)+(2+4z)zp'(z)+2p^2(z)+(10-4z)p(z)|<8$$

implies

$$|p(z)| < \frac{4}{3+\sqrt{5}}.$$

If $n = 2$, $A(z) = 6$, $B(z) = 8 - 2z$, $C(z) = -4$, $D(z) = 4 + 2z$, $M = 5$, $N = \frac{1}{2}$.

Let this case from Corollary 1 we deduce

Example 2. If $p \in \mathcal{H}[0, 2]$ then

$$|6z^2p'(z) + (8 - 2z)zp'(z) - 4p^2(z) + (4 + 2z)p(z)| < 5$$

implies

$$|p(z)| < \frac{1}{2}.$$

If $A(z) = A > 0$ then the Theorem can be rewritten as follows:

Corollary 2. *Let $M > 0$, $N > 0$ and let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy*

(i) $\mathrm{Re}\, B(z) \geq -nA$, $A > 0$,

(ii) $\mathrm{Re}\, [B(z) + D(z)] \geq \dfrac{M + N^2|C(z)|}{N}$.

If $p \in \mathcal{H}[0, n]$ and

$$|Az^2p'(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

then

$$|p(z)| < N.$$

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Gh. Oros and Georgia Irina Oros, *On a particular first-order nonlinear differential subordination I* (submitted).

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