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On a differential inequality II

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Abstract

We find conditions on the complex-valued function A defined in the unit disc U and the real constants α, β, γ, such that the differential inequality

$$\mathrm{Re}\ [A(z)p^2(z) - \alpha(zp'(z))^2 + \beta zp'(z) + \gamma] > 0$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U],\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots,\ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [1, p. 35].

Lemma A. [1, p. 35] *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ a function which satisfies*

$$\text{Re } \psi(\rho i, \sigma; z) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$, $z \in U$ and $n \geq 1$.

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } \psi(p(z), zp'(z); z) > 0$$

then

$$\text{Re } p(z) > 0.$$

2 Main results

Theorem. *Let $\alpha \geq 0$, $\beta \geq 0$, $\gamma \leq \frac{\alpha n^2}{4} + \frac{\beta n}{2}$ and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies*

$$\text{Re } A(z) \geq -\frac{\alpha n^2}{2} - \frac{\beta n}{2}. \tag{1}$$

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } [A(z)p^2(z) - \alpha(zp'(z))^2 + \beta zp'(z) + \gamma] > 0, \tag{2}$$

then

$$\text{Re } p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) - \alpha(zp'(z))^2 + \beta zp'(z) + \gamma. \tag{3}$$

From (2) we have

$$\text{Re } \psi(p(z), zp'(z); z) > 0, \text{ for } z \in U. \tag{4}$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, hence $-\sigma^2 \leq -\frac{n^2}{4}(1+\rho^2)^2$, and $z \in U$, by using (1) we obtain:

$$\text{Re } \psi(\rho i, \sigma; z) = \text{Re } [A(z)(\rho i)^2 - \alpha\sigma^2 + \beta\sigma + \gamma] =$$

$$= -\rho^2 \text{Re } A(z) - \alpha\sigma^2 + \beta\sigma + \gamma \leq$$

$$= -\rho^2 \text{Re } A(z) - \frac{\alpha n^2}{4}(1+\rho^2)^2 - \frac{\beta n}{2}(1+\rho^2) + \gamma \leq$$

$$\leq -\rho^2 \text{Re } A(z) - \frac{\alpha n^2}{4} - \frac{\alpha n^2}{2}\rho^2 - \frac{\alpha n^2}{4}\rho^4 - \frac{\beta n}{2} - \frac{\beta n}{2}\rho^2 \leq$$

$$\leq -\frac{\alpha n^2}{4}\rho^4 - \rho^2\left[\text{Re } A(z) + \frac{\alpha n^2}{2} + \frac{\beta n}{2}\right] - \frac{\alpha n^2}{4} - \frac{\beta n}{2} + \gamma \leq 0.$$

By using Lemma A we have that Re $p(z) > 0$.

If $p = 0$ then we obtain the Theorem from [2].

If $\gamma = \frac{\alpha n^2}{4} + \frac{\beta n}{2}$, Theorem can be rewritten as follows:

Corollary. *Let $\alpha \geq 0$, $\beta \geq 0$ and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies*

$$\text{Re } A(z) \geq -\frac{\alpha n^2}{2} - \frac{\beta n}{2}.$$

If $p \in \mathcal{H}[1, n]$ and

$$\text{Re } \left[A(z)p^2(z) - \alpha(zp'(z))^2 + \beta zp'(z) + \frac{\alpha n^2}{4} + \frac{\beta n}{2}\right] > 0$$

then

$$\text{Re } p(z) > 0.$$

If $\beta = 0$ then we obtain the Corollary from [2].

If $\alpha = \frac{1}{2}$, $n = 1$, $\beta = 3$, $A(z) = 1 + z$. In this case from Corollary we deduce

Example 1. If $p \in \mathcal{H}[1, 1]$ then

$$\text{Re } \left[(1 + 2z)p^2(z) - \frac{1}{2}(zp'(z))^2 + 3zp'(z) + \frac{13}{8}\right] > 0$$

implies

$$\text{Re } p(z) > 0.$$

If $\alpha = \frac{1}{2}$, $n = 2$, $A(z) = 1 + 2z$. In this case from Corollary 1 we deduce

Example 2. If $p \in \mathcal{H}[1, 2]$ then

$$\text{Re } \left[(1 + 3z)p^2(z) - 2(zp'(z))^2 + \frac{1}{4}zp'(z) + \frac{9}{4}\right] > 0$$

implies Re $p(z) > 0$.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Georgia Irina Oros, *On a differential inequality I*, (submitted)

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