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Posinormality versus hyponormality for Cesàro operators

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

The aim of this paper is the study of a relation between posinormality operators and hyponormality operators. It has been proved that posinormality does not imply hyponormality [9], but properties of Cesàro matrix and the unilateral shift suggest the plausibility of the reverse implication.

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1 Introduction

In this paper we study the properties of a large subclass of $B(\mathcal{H})$, the set of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$. We refer to $T^*T - TT^*$ as the **self - commutator** of T, denoted $[T^*, T]$. A self - adjoint operator P is **positive** if $< Pf, f > \geq 0$ for all $f \in \mathcal{H}$; the operator T is **normal** if $[T^*, T] = 0$ and T is hyponormal if $[T^*, T]$ is positive. When T^* is hyponormal, we say T is **cohyponormal**; T is **seminormal** if T is hyponormal or cohyponormal. If T is the restriction of a normal operator to an invariant subspace, then T is **subnormal**.

If $A \in \mathcal{B}(\mathcal{H})$ is to belong to our class, then A must not be "too far" from normal; more precisely, there must exist an **interrupter** $S \in \mathcal{B}(\mathcal{H})$ such that $AA^* = A^*SA$, or equivalently, $[A^*, A] = A^*(I - S)A$.

Two observations suggest the additional requirement that S be self - adjoint, even positive: (1) since AA^* in self - adjoint, each operator A is our subclass must satisfy $A^*S^*A = A^*SA$;

(2) since $< SAf, Af > = < A^*SAf, f > = ||A^*f||^2$ for all f, the interrupter S must be positive on Ran A (the range of A).

If the posinormal operator A is nonzero, the associated interrupted P must satisfy the condition $||P|| \geq 1$ since $||A||^2 = ||AA^*|| =$
$= ||A^*PA|| \leq ||A^*|| \cdot ||P|| \cdot ||A|| = ||P|| \cdot ||A||^2$.

Theorem 1.1. *If A is posinormal with interrupter P and A has dense range, then P is unique.*

Proof. See [10].

2 Examples

The example which motivated this motivated study is the Cesàro matrix

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

regarded as an operator on $\mathcal{H} = l^2$. The standard orthonormal basis on l^2 will be denoted by $\{e_n : n = 0, 1, 2, ...\}$. If D is the diagonal operator with diagonal $\left\{\frac{n+1}{n+2} : n = 0, 1, 2, ...\right\}$, then a routine computation verifies that

$$C_1^* D C_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = C_1 C_1^*$$

So the Cesàro operator on (l^2) is posinormal with interrupter D. C_1 is known to be hyponormal, even subnormal (see [4]). In [1], C_1 is shown to be hyponormal by looking at determinants of finite sections of $[C_1^*, C_1]$. We include here a brief and different proof - one that takes advantage of the availability of D.

Theorem 2.1. C_1 *is hyponormal.*

Proof. Since $I - D$ is a positive operator, we have

$$< [C_1^*, C_1]f, f >=< (I - D)C_1 f, C_1 f >\geq 0$$

for all f.

We have, in the Cesàro operator, an example of a nonnormal posinormal operator. The next proposition provides us with a large supply of additional examples, including the unilateral shift U.

Propozition 2.1. *Every unilateral weighted shift with nonzero weights is posinormal.*

Proof. See [10].

It is easy to see that if A is the unilateral weighted shift with weights w_k, then $[A^*, A]$, is the diagonal matrix with diagonal entries $\{w_0^2, w_1^2 - w_0^2, w_2^2 - w_1^2, ...\}$. If $\{w_k\}$ is increasing, then A is hyponormal. The special case when $w_0 = 2$ and $w_k = 1$ for all $k \geq 1$ provides an example of a posinormal operator that is neither hyponormal nor cohyponormal.

3 Posinormality versus hyponormality

The next result, from [2], will help settle the question (see Corollary 3.1) about the relation posinormality - hyponormality.

Theorem (Douglas) For $A, B \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:

(1) Ran A $\subseteq$ Ran B

(2) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$; and

(3) there exists a $T \in \mathcal{B}(\mathcal{H})$ such that $A = BT$.

Moreover, if (1), (2) and (3) hold, then there is an unique operator T such that:

(a) $||T||^2 = \inf\{\mu | AA^* \leq \mu BB^*\}$;

(b) Ker A = Ker T.

We know that a hyponormal operator T must satisfy the inequality $||T^* f|| \subseteq ||Tf||$ for all f. Statement (a) of the following proposition gives us an analogous result for posinormal operators; this result, together with the above theorem of Douglas, will lead to a characterization of posinormality (see Theorem 3.1).

Propozition 3.1. *If A is posinormal with (positive) interrupter P, then the following statements hold:*

(a) $||A^* f|| = ||\sqrt{P} A f|| \leq ||\sqrt{P}|| \cdot ||Af||$ *for every f in* $\mathcal{H}$

(b) $||\sqrt{P} A|| = ||A||$.

Proof. (a) Since A is posinormal and P is positive

$$||A^* f||^2 =< AA^* f, f >=< A^* PAf, f >= ||\sqrt{P} Af||^2 \leq ||\sqrt{P}||^2 \cdot ||Af||^2$$

for all f in $\mathcal{H}$.

(b) From (a) we see that $||A^*|| = ||\sqrt{P} A||$, and $||A|| = ||A^*||$ is universal.

We note that if A is posinormal, the condition (2) in the theorem above is satisfied with $\lambda = ||\sqrt{P}||$ and $B = A^*$. If condition (3) in the theorem holds, then there is an operator $T \in \mathcal{B}(\mathcal{H})$ such that $A = A^* T$, so $A^* = T^* A$;

consequently, A is posinormal with interrupter TT^*. Thus Douglas theorem has led almost immediately to the following result.

Theorem 3.1. *For $A \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:*

(1) A is posinormal;

(2) Ran $A \leq$ Ran A^*;

(3) $AA^* \leq \lambda^2 A^*A$ for some $\lambda \geq 0$; and

(4) there exists a $T \in \mathcal{B}(\mathcal{H})$ such that $A = A^*T$.

Moreover if (1), (2), (3), and (4) hold, then there is an unique operator T such that:

a) $||T^2|| = \inf\{\mu | AA^* \leq \mu A^*A\}$;

b) Ker A = Ker T.

Corollary 3.1. *Every hyponormal operator is posinormal.*

Proof. If a is hyponormal, the condition (3) is satisfied with $\lambda = 1$.

Let $[A] = \{TA : T \in \mathcal{B}(\mathcal{H})\}$, the left ideal in $\mathcal{B}(\mathcal{H})$ generated by A. If A is posinormal, then, because of (4), we have $A^* = T^*A$ for some bounded operator T, so $A^* \in [A]$. Conversely, if $A^* \in [A]$, then $A^* = kA$ for some $k \in \mathcal{B}(\mathcal{H})$, so A is posinormal with interrupter $P = k^*R$. In summary, we have the following corollary.

Corollary 3.2. *A is posinormal if and only if $A^* \in [A]$.*

We note that if A is hyponormal, then for some contraction $k, A^* = kA$ (see [10], p. 3). A straight forward computation shows that in the case of the Cesàro operator the contraction $k = k(C_1)$ takes from $k(C_1) = (k_{mn})$ where

$$k_{mn} = \begin{cases} \dfrac{1}{n+2}, & \text{if } \ m \le n \\ -\dfrac{n+1}{n+2}, & \text{if } \ m = n+1 \\ 0, & \text{if } \ m > n+1. \end{cases}$$

It is not hard to verify that $k(C_1)^* \cdot k(C_1) = D$.

While the Cesàro matrix C_1 is hyponormal, the remaining p-Cesàro matrices:

$$C_p = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \left(\frac{1}{2}\right)^p & \left(\frac{1}{2}\right)^p & 0 & 0 & \cdots \\ \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $p > 1$ are not (see [7]) there will use Corollary 3.2 to show that all of these operators are, however, posinormal. Define $B_p = (b_{mn})$ by

$$b_{mn} = \begin{cases} 1 - \left(\dfrac{n+1}{n+2}\right)^p, & \text{if } \ m \le n \\ -\left(\dfrac{n+1}{n+2}\right)^p, & \text{if } \ m = n+1 \\ 0, & \text{if } \ m > n+1. \end{cases}$$

We observe that $B_1 = k(C_1)$. To see that B_p is bounded when $p > 1$, we note that this matrix can be decomposed as $B_p = Y + Z$ where $Y = (y_{mn})$ satisfies $y_{mn} = b_{mn}$ when $m = n+1$ and $y_{mn} = 0$ otherwise (so Y is a weighted shift) and Z is the upper triangular matrix whose entries on and above the main diagonal agree with those form B_p and whose other entries are all zero. We note that the entries of Z are all nonnegative. Since

$1 - \dfrac{(n+1)^p}{(n+2)^p} < \dfrac{p}{n+2}$ for all $p > 1$ (see [3, Theorem 42, 2.15.3, page 40]), Z is entrywise dominated by pC_1^*, an operator known to be bounded; Y is clearly a bounded operator, and consequently B_p is also bounded and $||B_p|| \leq ||Y|| + ||Z|| \leq 1 + 2p$. A routine computation gives $C_p^* = B_p C_p$, and the following theorem has been proved.

Theorem 3.2. *C_p is posinormal for all $p \geq 1$.*

We have seen that C_1 is posinormal, but what about C_1^*? Corollary 3.2 will help us here also, for it can be verified that $C_1 = BC_1^*$ when $B = C_1 - U^*$, so $C_1 \in [C_1^*]$; it can also be easily checked that $k(C)B = I = Bk(C)$. While B^*B is the interrupter for the posinormal operator C_1^*, the matrix product in the other order takes on a much simpler form; BB^* is the diagonal matrix with diagonal $\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...\right\}$. These observations justify the next theorem and its corollary.

Theorem 3.3. *C_1^* is posinormal with interrupter $P = B^*B = (C_1^* - U)(C_1 - U^*)$.*

Corollary 3.3. $||C_1 - U^*|| = \sqrt{2}$.

4 Shift - conjugated Cesàro matrices.

In this section we consider the terraced matrix $T_{k+1} = (U^k)^* C_1 U^*$, where U is an unilateral shift, for positive integers k:

$$T_k = \begin{pmatrix} \frac{1}{k} & 0 & 0 & \cdots & \cdots \\ \frac{1}{k+1} & \frac{1}{k+1} & 0 & \cdots & \cdots \\ \frac{1}{k+2} & \frac{1}{k+2} & \frac{1}{k+2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Visually, T_{k+1} can be obtained from the Cesàro matrix C_1 by deleting the first k rows and columns from C_1. We note that in fact for all $k > 0$ (and not just the positive integers) the matrix T_k gives a bounded operator on l^2 : T_k can be expressed as $D_k C_1$ where D_k is the diagonal matrix with diagonal $\left\{\frac{1+n}{k+n} : n = 0, 1, 2, ...\right\}$, it is clear by inspection that $||T_k|| \le ||C_1|| = 2$ for $k \ge 1$ (the proof that $||C_1|| = 2$ appears in [1]), and for $0 < k < 1$, we have $||T_k|| = ||D_k C_1|| \le ||D_k|| \cdot ||C_1|| = \frac{2}{k}$. Results from [8] and [9] justify the remaining assertions of the next theorem.

Theorem 4.1. *For each $k > 0, T_k$ is a bounded operator on l^2; $||T_k|| = 2$ when $k \ge 1$ and $||T_k|| \le \frac{2}{k}$ when $0 < k < 1$.*

We show that, for all $k > 0, T_k$ is posinormal with interrupter $P = (p_{mn})$ whose entries are given by

$$p_{mn} = \begin{cases} \frac{n^2 + (2k+1)n + k^2 + 1}{(n+k+1)^2}, & \text{if } m = n \\ \frac{1-k}{(m+k+1)(n+k+1)}, & \text{if } m \ne n. \end{cases}$$

Note that when $k = 1, P$ reduces to the diagonal operator D. To see that P is bounded, we observe that P can be decomposed as $P = L + R + C^*$ where R is the diagonal matrix with diagonal from P and L is the lower triangular matrix whose entries below the main diagonal agree with those from P and whose other entries are all zero, then $||R|| \leq 1$ and $||L|| \leq |k-1| \cdot ||C_1|| = 2|k-1|$, so $||P|| \leq 1 + 4|k-1|$.

One can check that $PT_k = (\alpha_{mn})$ has matrix entries satisfying:

$$\alpha_{mn} = \begin{cases} \dfrac{n+1}{(m+k+1)(n+k)}, & \text{if } m \geq n \\ \dfrac{1-k}{(m+k+1)(n+k)}, & \text{if } m < n \end{cases};$$

using these entries, it is not hard to verify that $T_k T_k^* = T_k^* P T_k$. In order to see that T_k is posinormal, it remains to show that P is positive; it suffices to show that P_N, the N^{th} finite section of P; (involving rows $m = 0, 1, ..., N$, and columns $n = 0, 1, ..., N$), has positive determinant for each positive integer N. For columns $n = 1, 2, N$, we multiply the n^{th} column from P_N by $\dfrac{k+n+1}{k+n}$ and then substract from the $(n-1)^{st}$ column. Call the new matrix P'_N and note that $\det P'_N = \det P_N$. We now work with the rows of P'_N: For $m = 1, 2, ..., N$, we multiply the m^{th} row from P'_N by $\dfrac{k+m+1}{k+m}$ and then subtract from the $(n-1)^{st}$ row. The resulting matrix is tridiagonal and also has the same determinant as P_N; that new matrix is constantly -1 on the two off-diagonals and is almost constantly 2 on the main diagonal - the only exception is the last entry: $\dfrac{k^2 + 2NK + N^2 + N + 1}{(K+N+1)^2}$. To finish our computation, we work this tridiagonal matrix into triangular form: multiply each row $m = 0, 1, ..., N-1$ by $\dfrac{m+1}{m+2}$ and add to the $(m+1)^{st}$ row. The new matrix is triangular and has diagonal $\left\{2, \frac{3}{4}, \frac{4}{3}, ..., \frac{N+1}{N}, \frac{N+k^2+1}{(N+1)(N+k+1)^2}\right\}$; from this we conclude that $\det P_N = \frac{N+k^2+1}{(N+k+1)^2}$.

We note that the positivity (and uniqueness) of P could have been demonstrated more briefly using the fact that T_k has dense range; however, our computational procedure provides a springboard for investigating the positivity of $I - P$. To see when $I - P$ is positive, we compute $\det(I - P)_N$ where $(I - P)_N$ is the N^{th} finite section of $I - P$. Following exactly the same sequence of column and row operations we used for P_N, we arrive at a tridiagonal matrix of the following form:

$$\overline{Y}_N = \begin{pmatrix} d_0 & a_0 & 0 & \cdots & \cdots & 0 \\ a_0 & d_1 & a_1 & \cdots & \cdots & 0 \\ 0 & a_1 & d_2 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & d_{N-1} & a_{N-1} \\ 0 & 0 & \cdots & \cdots & a_{N-1} & d_N \end{pmatrix}$$

where $a_n = -\frac{1}{k+n+1}$, $d_n = \frac{2k+2n+3}{(k+n+1)^3}(0 \le n \le N-1)$, and $d_N = \frac{2k+N}{(N+k+1)^2}$. In transforming $\overline{Y}_N$ into a triangular matrix with the same determinant, we find that the new matrix has diagonal entries δ_n which are given by a recursion formula: $\delta_0 = d_0$, $\delta_n = d_n - \frac{a_{n-1}^2}{\delta_{n-1}}(1 \le n \le N)$. An induction argument shows that $\delta_n \ge \frac{n+k+2}{(n+k+1)^2}$ for $0 \le n \le N-1$; since d_N departs the pattern set by the earlier $d_n''s$, δ_n must be handled separately: $\delta_N = d_N - \frac{a_{N-1}^2}{N-1} \ge \frac{k-1}{(N+k+1)^2}$. So $\det(I-P)_N = \prod_{j=0}^{N} \delta_j > 0$ for $k > 1$.

The computation just completed tells us that T_k is hyponormal when

$K > 1$. Further calculations reveal an exact value for the determinant (we omit the details):

$$\det(I-P)_N = \left[\prod_{j=0}^{N} \frac{1}{j+k+1}\right] \left[(k-1)\sum_{j=0}^{N-1} \frac{1}{j+k+1} + \frac{2k+N}{N+k+1}\right].$$

For $k < 1$, $\det(I-P)_N$ is eventually negative, so T_k is not hyponormal in this case. We summarize the main results in the following theorem.

Theorem 4.2. *T_k is posinormal for all $k > 0$; T_k is hyponormal if and only if $k \geq 1$.*

5 Discrete Cesàro operator C_1

In this brief section we consider the lower triangular matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

regarded as operators on l^2. These operators have been studied in [5,6]. Define $B = (b_{mn})$ by

$$b_{mn} = \begin{cases} \frac{1}{n+2}, & \text{if } m \leq n \\ -\frac{n+1}{n+2}, & \text{if } m = n+1 \\ 0, & \text{if } m > n+1. \end{cases}$$

We note that B is the contraction (hence bounded) operator $k(C_1)$ from section 2. A routine computation gives $C_1^* = BC_1$, settling the question of posinormality for C_1.

Theorem 5.1. *C_1 is posinormal.*

References

[1] A. Brown, P.R. Halmos, A.L. Shiels, *Cesàro Operators*, Acta Sci. Math. Szeged, 26 (1965), 125 - 137.

[2] R.G. Douglas, *On Majorization, Factorization, and Range Inclusion of Operators on Hilbert Spaces*, Proc. Amer. Math. Soc., 17 (1966), 413 - 415.

[3] G. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Second Edition, Cambridge University Press, Cambridge, 1989.

[4] T.L. Kriete III, D. Trutt, *The Cesàro Operator in l^2 is Subnormal*, Amer. J. Math., 93 (1971), 215 - 225.

[5] H.C. Rhaly, Jr., *Discrete Generalized Cesàro Operators*, Proc. Amer. Math. Soc., 86 (1982), 405 - 409.

[6] H.C. Rhaly, Jr., *Generalized Cesàro Matrices*, Canad. Math. Bull., 27 (1984), 417 - 422.

[7] H.C. Rhaly, Jr., *p-Cesàro Matrices*, Houston J. Math., 15 (1989), No.1, 137 - 146.

[8] H.C. Rhaly, Jr., *Terraced Matrices*, Bull. London Math. Soc., 21 (1989), 399 - 406.

[9] H.C. Rhaly, Jr., *Posinormal operators*, J. Math. Soc. Japan, vol. 46, No. 4, 1994, 587 - 605.

[10] B.E. Rhodes, *Generalized Haussdorf Matrices Bounded on l^p and c*, Acta Sci. Math. Szeged, 43 (1981), 333 - 345.

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