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# On a subclass of functions with negative coefficients

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Dedicated to Professor dr. Gheorghe Micula on his $60^{th}$ birthday

**Abstract**

We determine conditions for a function to be n-close to convex of order $\alpha$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with negative coefficients.

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## 1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In ([4]) the subfamily $T$ of $S$ consisting of functions $f$ of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ \ a_j \geq 0, j = 2, 3, ...,\ \ z \in U \tag{1}$$

was introduced.

The purpose of this paper is to give a condition for $f \in T$ to be n-close to convex of order $\alpha$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, and to determine some properties of this class.

## 2 Preliminary results

Let $D^n$ be the Sălăgean differential operator (see [2]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

**Remark 2.1.** *If* $f \in T, f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then* $D^n f(z) = z - \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

**Theorem 2.1.[2].** *If* $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then the next assertions are equivalent:*

*(i)* $\sum\limits_{j=2}^{\infty} j a_j \leq 1$

*(ii)* $f \in T$

*(iii)* $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* $S^*$ *is the well-known class of starlike functions.*

**Definition 2.1.[2].** *Let* $\alpha \in [0,1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\alpha) = \left\{ f \in A : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

*is the set of n-starlike functions of order $\alpha$.*

**Remark 2.2.** *If $f \in S_n(\alpha)$ according to the definition of the Sălăgean differential operator we can write that*

$$Re\frac{z(D^n f(z))'}{D^n f(z)} > \alpha$$

*and thus the function $F(z) = D^n f(z) \in S(\alpha)$, $\alpha \in [0, 1)$, where*

$$S(\alpha) = \left\{h \in A : Re\frac{zh'(z)}{h(z)} > \alpha,\ z \in U\right\}.$$

**Definition 2.2.[2].** $T_n(\alpha) = T \bigcap S_n(\alpha)$.

**Definition 2.3.[3].** *Let $\alpha \in [0, 1), \beta \in (0, 1]$ and let $n \in \mathbb{N}$; we define the class $T_n(\alpha, \beta)$ of n-starlike functions of order $\alpha$ and type $\beta$ with negative coefficients by*

$$T_n(\alpha, \beta) = \{f \in A : |J_n(f, \alpha; z)| < \beta, z \in U\},$$

*where*

$$J_n(f, \alpha; z) = \frac{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha},\ \ z \in U$$

**Remark 2.3.** *The class $T_0(\alpha, 1)$ is the class of starlike functions of order $\alpha$ with negative coefficients; $T_1(\alpha, 1)$ is the well-known class of convex functions of order $\alpha$ with negative coefficients; $T_n(\alpha, 1)$ is the class of n-starlike functions of order $\alpha$ with negative coefficients i.e. $T_n(\alpha, 1) = T \bigcap S_n(\alpha)$. We also note that the functions in $T_n(\alpha, \beta)$ are univalent because $T_n(\alpha, \beta) \subset T_n(\alpha, 1)$, $\beta \in (0, 1)$ and $T_n(\alpha_1, \beta) \subset T_n(\alpha, \beta)$ with $1 > \alpha_1 > \alpha \geq 0$, $\beta \in (0, 1]$.*

**Theorem 2.2.[3].** *Let $\alpha \in [0,1), \beta \in (0,1]$ and $n \in \mathbb{N}$. The function $f$ of the form (1) is in $T_n(\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j-1+\beta(j+1-2\alpha)]a_j \le 2\beta(1-\alpha)$$

*The result is sharp and the extremal functions are:*

$$f_j(z) = z - \frac{2\beta(1-\alpha)}{j^n[j-1+\beta(j+1-2\alpha)]} z^j, j = 2, 3, ....$$

*From this result we have $T_{n+1}(\alpha,\beta) \subset T_n(\alpha,\beta)$, $n \in \mathbb{N}$.*

**Definition 2.4.[3].** *Let $I_c : A \to A$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1,\infty)$, $F \in A$ and*

(2) $$f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} F(t) dt.$$

*We note if $F \in A$ is a function of the form (1), then*

(3) $$f(z) = I_c F(z) = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j.$$

**Remark 2.4.** *In [3] is showed that if $F \in T_n(\alpha,\beta)$ then $f = I_c(F) \in T_n(\alpha,\beta)$.*

**Definition 2.5.[1].** *Let $f \in A$. We say that $f$ is n-close to convex of order $\alpha$ with respect to a half-plane, and denote by $CC_n(\alpha)$ the set of these functions, if there exists $g \in S_n(0) = S_n$ so that*

$$Re \frac{D^{n+1} f(z)}{D^n g(z)} > \alpha, \; z \in U,$$

*where $n \in \mathbb{N}, \alpha \in [0,1)$.*

**Remark 2.5.** *$CC_0(\alpha) = CC(\alpha)$, where $CC(\alpha)$ is the well-known class of close to convex functions of order $\alpha$.*

**Remark 2.6.** *In [1] the author show that if $n \in \mathbb{N}$ and $\alpha \in [0,1)$ then $CC_{n+1}(\alpha) \subset CC_n(\alpha)$ and thus the functions from $CC_n(\alpha)$ are univalent.*

**Remark 2.7.** *From Remark 2.3 and Theorem 2.2 we have for $f$ of the form (1) with $f \in T_n(\alpha, 1) = T_n(\alpha)$:*

$$\sum_{j=2}^{\infty} j^n(j-\alpha)a_j \leq 1-\alpha, \text{ where } \alpha \in [0,1)$$

# 3 Main results

**Definition 3.1.** *Let $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$. We say that $f$ is in the class $CCT_n(\alpha), \alpha \in [0,1)$, $n \in \mathbb{N}$, with respect to the function $g \in T_n(0)$, if:*

$$Re\frac{D^{n+1}f}{D^n g} > \alpha, \ z \in U.$$

**Theorem 3.1.** *Let $\alpha \in [0,1)$ and $n \in \mathbb{N}$. The function $f \in T$ of the form (1) is in $CCT_n(\alpha)$, with respect to the function $g \in T_n(0)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, ...$, if and only if*

$$\sum_{j=2}^{\infty} j^n[ja_j + (2-\alpha)b_j] < 1-\alpha \tag{4}$$

**Proof.** Let $f \in CCT_n(\alpha)$, with $\alpha \in [0,1)$. We have

$$Re\frac{D^{n+1}f(z)}{D^n g(z)} > \alpha.$$

If we take $z \in [0,1)$, we have (see Remark 2.1):

$$\frac{1-\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{1-\sum_{j=2}^{\infty} j^{n} b_j z^{j-1}} > \alpha \tag{5}$$

From $g \in T_n(0) = T_n(0,1)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j, b_j \geq 0, j = 2,3,...,$ we have (see Remark 2.7):

$$\sum_{j=2}^{\infty} j^{n+1} b_j \leq 1. \tag{6}$$

We have: $\sum_{j=2}^{\infty} j^n b_j z^{j-1} \leq \sum_{j=2}^{\infty} j^{n+1} b_j z^{j-1} < \sum_{j=2}^{\infty} j^{n+1} b_j$.

From (6) we obtain: $\sum_{j=2}^{\infty} j^n b_j z^{j-1} < 1$ and thus $1 - \sum_{j=2}^{\infty} j^n b_j z^{j-1} > 0$.

In this condition from (5) we obtain:

$$1-\sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1} > \alpha \left[1-\sum_{j=2}^{\infty} j^{n} b_j z^{j-1}\right]$$

Letting $z \to 1^-$ along the real axis we have:

$$1-\sum_{j=2}^{\infty} j^{n+1} a_j > \alpha - \sum_{j=2}^{\infty} j^{n} \alpha b_j,$$

and thus:

$$\sum_{j=2}^{\infty} j^n [j a_j - \alpha b_j] < 1-\alpha.$$

From $\sum_{j=2}^{\infty} j^n [j a_j + (2-\alpha) b_j] > \sum_{j=2}^{\infty} j^n [j a_j - \alpha b_j]$ we have that from

$$\sum_{j=2}^{\infty} j^n [j a_j + (2-\alpha) b_j] < 1-\alpha \tag{7}$$

we obtain $Re\dfrac{D^{n+1}f(z)}{D^n g(z)} > \alpha$.

Now let take $f \in T$ and $g \in T_n(0)$ for which the relation (4) hold.

The condition $Re\dfrac{D^{n+1}f(z)}{D^n f(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right) < 1 \tag{8}$$

We have

$$\alpha - Re\left(\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right) \le \alpha + \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| =$$

$$= \alpha + \left|\frac{1 - \sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}}{1 - \sum_{j=2}^{\infty} j^n b_j z^{j-1}} - 1\right| \le \alpha + \frac{\sum_{j=2}^{\infty} j^n |b_j - ja_j| \cdot |z|^{j-1}}{1 - \sum_{j=2}^{\infty} j^n b_j |z|^{j-1}} \le$$

$$\le \alpha + \frac{\sum_{j=2}^{\infty} j^n |b_j - ja_j|}{1 - \sum_{j=2}^{\infty} j^n b_j} \le \alpha + \frac{\sum_{j=2}^{\infty} j^n (b_j + ja_j)}{1 - \sum_{j=2}^{\infty} j^n b_j} =$$

$$= \frac{\alpha + \sum_{j=2}^{\infty} j^n [ja_j + (1-\alpha)b_j]}{1 - \sum_{j=2}^{\infty} j^n b_j}$$

Using (8) we obtain:

$$\alpha + \sum_{j=2}^{\infty} j^n [ja_j + (2-\alpha)b_j] < 1$$

that is the condition (4).

**Remark 3.1.** *If we take $f \equiv g$ we obtain from Theorem 3.1*

$$\sum_{j=2}^{\infty} j^n a_j [j a_j + 2 - \alpha] < 1 - \alpha$$

*From* $\sum_{j=2}^{\infty} j^n a_j [j + 2 - \alpha] > \sum_{j=2}^{\infty} j a_j (j - \alpha)$ *we obtain:*

$$\sum_{j=2}^{\infty} j a_j (j - \alpha) < 1 - \alpha$$

*Thus we obtain the result from Remark 2.7.*

**Remark 3.2.** *From the proof of the Theorem 3.1 we obtain a necessary condition for a function $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$ to be in the class $CCT_n(\alpha)$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with respect to the function $g \in T_n(0)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$:*

$$\sum_{j=2}^{\infty} j^n [j a_j - \alpha b_j] < 1 - \alpha.$$

**Theorem 3.2.** *If $F \in CCT_n(\alpha), \alpha \in [0,1), n \in \mathbb{N}$, with respect to the function $G \in T_n(0)$ and $f = I_c(F)$, $g = I_c(F)$ where $I_c$ is defined by (2), then $f \in CCT_n(\alpha)$ with respect to the function $g \in T_n(0)$ (see Remark 2.4)*

**Proof.** From $F(z) = z - \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2, 3, ...$ and $f(z) = I_c(F)(z)$ we have (see (3)):

$$f(z) = z - \sum_{j=2}^{\infty} \alpha_j z^j, \text{ where } \alpha_j = \frac{c+1}{c+j} a_j, \ j = 2, 3, ...$$

From $G(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j, b_j \geq 0, j = 2, 3, ...$ and $g(z) = I_c(G)(z)$ we have:

$$g(z) = z - \sum_{j=2}^{\infty} \beta_j z^j, \text{ where } \beta_j = \frac{c+1}{c+j} b_j, \ j = 2, 3, ...$$

From $F \in CCT_n(\alpha)$ with respect to the function $G \in T_n(0)$ we have (see Theorem 3.1):

$$\sum_{j=2}^{\infty} j^n [j a_j + (2 - \alpha) b_j] < 1 - \alpha. \tag{9}$$

From Theorem 3.1 we need only to show that:

$$\sum_{j=2}^{\infty} j^n [j \alpha_j + (2 - \alpha) \beta_j] < 1 - \alpha.$$

We have for $c \in (-1, \infty)$ and $j = 2, 3, ...$:

$$\sum_{j=2}^{\infty} j^n [j \alpha_j + (2 - \alpha) \beta_j] =$$

$$= \sum_{j=2}^{\infty} \frac{c+1}{c+j} j^n [j a_j + (2 - \alpha) b_j] < \sum_{j=2}^{\infty} j^n [j a_j + (2 - \alpha) b_j]$$

From (9) we have:

$$\sum_{j=2}^{\infty} j^n [j \alpha_j + (2 - \alpha) \beta_j] < 1 - \alpha.$$

# References

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