General Mathematics Vol. 11, No. 1–2 (2003), 77–86

Data dependence theorems for operators on cartesian product spaces

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

We present some abstract data dependence theorems of the fixed point set for operators $f, g : X \times Y \to X \times Y$, using the c-Picard operators technique.

2000 Mathematical Subject Classification: 47H10.

Keywords: Picard operators, c-Picard operators, data dependence.

1 Picard operators, c-Picard operators

In this section we present some definition useful in the next part of the paper.

Definition 1.1.(I.A. Rus [4]). *Let (X,d) be a metric space. An operator* $A : X \to X$ *is (uniformly) Picard operator (PO) if exists* $x^* \in X$ *such that:*

(a) $F_A = \{x^*\}$,

(b) $(A^n(x))_{n\in N}$ *converges (uniformly) to* x^**, for all* $x \in X$.

Definition 1.2.(I.A. Rus [4]). *Let (X,d) be a metric space. An operator* $A : X \to X$ *is (uniformly) weakly Picard operator (WPO) if:*

(a) the sequence $(A^n(x))_{n\in N}$ *converges (uniformly), for all* $x \in X$,

(b) the limit (which may depend on x) is a fixed point of A.

If A is weakly Picard operator then we consider the following operator:

$$\begin{aligned} A^{\infty} &: X \to X, \\ A^{\infty}(x) &= \lim_{n\to\infty} A^n(x). \end{aligned} \tag{1}$$

Definition 1.3.(I.A. Rus [2]). *Let (X,d) be a metric space. An operator* $A : X \to X$ *is c-(uniformly) weakly Picard operator (c-WPO) if:*

(a) A is (uniformly) weakly Picard;

(b) exists $c > 0$ *such that.:*

$$d\left(x, A^{\infty}(x)\right) \leq c \cdot d\left(x, A(x)\right), \tag{2}$$

for all $x \in X$.

Example 1.1. *Let (X,d) be a complete metric space and an operator* $A : X \to X$ *such that:*

$$d(A(x), A(y)) \leq \alpha_1 d(x,y) + \alpha_2 d(x, A(x)) + \alpha_3 d(y, A(y)) +$$

$$+ \alpha_4 d(x, A(y)) + \alpha_5 d(y, A(x)),$$

with $\alpha_i > 0$, $i = \overline{1,5}$, $\alpha_1 + \ldots + \alpha_4 + 2\alpha_5 < 1$, *for all* $x, y \in X$. *Then A is c-Picard operator with* $c = \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_5}$.

Example 1.2.(L.B. Ćirić [1]). *Let (X,d) be a complete metric space and an operator $A : X \to X$ such that:*

$$d(A(x), A(y)) \leq$$

$$\leq \alpha \cdot \max \left\{ d(x,y), d(x,A(x)), d(y,A(y)), \frac{1}{2}\left[d(x,A(y)) + d(y,A(x))\right] \right\},$$

with $\alpha \in [0;1[$, for all $x, y \in X$. Then A is c-Picard operator with $c = \dfrac{1}{1-\alpha}$.

For other examples of c-Picard operators see S. Mureşan, I.A. Rus [3], I.A.Rus [4].

An important data dependence result which is used in our paper is the following:

Theorem 1.1.(I.A.Rus, S. Mureşan [3]). *Let (X, d) be a metric space. and $A_1, A_2 : X \to X$ two operator such that.:*

(i) A_i is c_i-WPO, $i = \{1, 2\}$;

(ii) exists $\eta > 0$ such. that.: $d\left(A_1(x), A_2(x)\right) \leq \eta$, for all $x \in X$

Then:

$$H\left(F_{A_1}, F_{A_2}\right) \leq \eta \cdot \max\left\{c_1, c_2\right\}, \tag{3}$$

where H is Hausdorff-Pompeiu metric on $P(X)$.

2 Fixed point theorems

In this section we present some fixed point theorems for operators $f : X \times Y \to X \times Y$, where X, Y are metric spaces.

Theorem 2.1. *Let (X, d), (Y, ρ) be two complete metric spaces and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. Suppose there exist $\psi_1, \psi_2 : \mathbb{R}_+^6 \to \mathbb{R}_+$ continuous functions such that:*

$$\text{(i)}\quad \begin{aligned} d(f_1(x_1,y_1),f_1(x_2,y_2)) \leq \psi_1(& d(x_1,x_2),\rho(y_1,y_2),d(x_1,f_1(x_1,y_1)),\\ & d(x_2,f_1(x_2,y_2)),d(x_1,f_1(x_2,y_2)),\\ & d(x_2,f_1(x_1,y_1))\,),\end{aligned}$$

$(x_1,y_1),\ (x_2,y_2)\in X\times Y$;

$$\text{(ii)}\quad \begin{aligned} \rho(f_2(x_1,y_1),f_2(x_2,y_2)) \leq \psi_2(& d(x_1,x_2),\rho(y_1,y_2),\rho(y_1,f_2(x_1,y_1)),\\ & \rho(y_2,f_2(x_2,y_2)),\rho(y_1,f_2(x_2,y_2)),\\ & \rho(y_2,f_2(x_1,y_1))\,),\end{aligned}$$

$(x_1,y_1),\ (x_2,y_2)\in X\times Y$;

(iii) *for any* $t_1,t_2\in\mathbb{R}^6_+$ *such that* $t_1\leq t_2$ *we have* $\psi_i(t_1)\leq\psi_i(t_2)$, $i=\overline{1,2}$;

(iv) $\psi_i(t_1+t_2)\leq\psi_i(t_1)+\psi_i(t_2)$, *for all* $t_1,t_2\in\mathbb{R}^6_+$, $i=\overline{1,2}$;

(v) *for any* $\lambda\in\mathbb{R}_+$ *we have* $\psi_i(\lambda t)\leq\lambda\psi_i(t)$, *for all* $t\in\mathbb{R}^6_+$, $i=\overline{1,2}$;

(vi) $\psi_1(0,0,0,1,1,0)<1$ *and* $\psi_1(1,0,0,1,1,0)<1$;

(vii) $\psi_2(0,0,0,1,1,0)<1$ *and* $\psi_2(0,1,0,1,1,0)<1$;

(viii) $$\frac{\psi_1(1,0,1,0,1,0)}{1-\psi_1(0,0,0,1,1,0)}<1;$$

(ix) $$\frac{\psi_2(0,1,1,0,1,0)}{1-\psi_2(0,0,0,1,1,0)}<1;$$

(x) $$\frac{\psi_1(0,1,0,0,0,0)}{1-\psi_1(1,0,0,0,1,1)}\cdot\frac{\psi_2(1,0,0,0,0,0)}{1-\psi_2(0,1,0,0,1,1)}<1.$$

In these conditions we have that $F_f=\{(x^*,y^*)\}$.

Proof. From conditions (i)-(ix) we obtain that $F_{f_1(\cdot,y)}=\{x^*(y)\}$ and $F_{f_2(x,\cdot)}=\{y^*(x)\}$ (see M.A. Şerban [5])

We define the following operators:

$$\begin{gathered} P:Y\to X\\ P(y)=x^*(y)\in F_{f_1(\cdot,y)}\end{gathered}\tag{4}$$

$$(5)\qquad \begin{aligned} &Q : X \to Y \\ &Q\,(x) = y^*(x) \in F_{f_2(x,\cdot)} \end{aligned}$$

It is easy to check that P and Q are lipschitz:

$$(6)\qquad d(P(y_1), P(y_2)) \leq \frac{\psi_1(0,1,0,0,0,0)}{1-\psi_1(1,0,0,0,1,1)} \cdot \rho(y_1, y_2),$$

$$(7)\qquad \rho(Q(x_1), Q(x_2)) \leq \frac{\psi_2(1,0,0,0,0,0)}{1-\psi_2(0,1,0,0,1,1)} \cdot d(x_1, x_2),$$

which implies that $P \circ Q$ is contraction on X, therefore we have that $F_{P\circ Q} = \{x^*\}$ and $(x^*, Q\,(x^*)) \in F_f$. The uniqueness of fixed point for f is obtained from the uniqueness of x^* as a fixed point for $P \circ Q$.

Using this general result we obtain the following fixed point theorems.

Corollary 2.1. *Let (X, d), (Y, ρ) be two complete metric spaces and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. Suppose that:*

(i)
$$\begin{aligned} d(f_1(x_1,y_1), f_1(x_2,y_2)) \;&\leq \alpha_1\, d(x_1,x_2) + \alpha_2\rho(y_1,y_2) + \\ &+\alpha_3 d(x_1, f_1(x_1,y_1)) + \alpha_4 d(x_2, f_1(x_2,y_2)) + \\ &+\alpha_5 d(x_1, f_1(x_2,y_2)) + \alpha_6 d(x_2, f_1(x_1,y_1))\,), \end{aligned}$$
$(x_1, y_1),\ (x_2, y_2) \in X \times Y$;

(ii)
$$\begin{aligned} \rho(f_2(x_1,y_1), f_2(x_2,y_2)) \;&\leq \beta_1\, d(x_1,x_2) + \beta_2\rho(y_1,y_2) + \\ &+\beta_3\rho(y_1, f_2(x_1,y_1)) + \beta_4\rho(y_2, f_2(x_2,y_2)) + \\ &+\beta_5\rho(y_1, f_2(x_2,y_2)) + \beta_6\rho(y_2, f_2(x_1,y_1))\,), \end{aligned}$$
$(x_1, y_1),\ (x_2, y_2) \in X \times Y$;

(iii) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 < 1,\ \alpha_i \in \mathbb{R}_+,\ i = \overline{1,6}$*;*

(iv) $\beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 + \beta_6 < 1,\ \beta_i \in \mathbb{R}_+,\ i = \overline{1,6}$.

In these conditions we have $F_f = \{(x^*, y^*)\}$.

Proof. We'll apply Theorem 2.1 for

$$\begin{aligned}
\psi_1(r_1,r_2,r_3,r_4,r_5,r_6) &= \sum_{i=1}^{6}\alpha_i\cdot r_i,\\
\psi_2(r_1,r_2,r_3,r_4,r_5,r_6) &= \sum_{i=1}^{6}\beta_i\cdot r_i.
\end{aligned}$$

Conditions (iii)-(x) are easy to check.

Corollary 2.2. *Let* (X,d)*,* (Y,ρ) *be two complete metric spaces and* $f: X\times Y\to X\times Y$*,* $f=(f_1,f_2)$*. Suppose that:*

(i)
$$\begin{aligned}
d(f_1(x_1,y_1),f_1(x_2,y_2)) \quad & \alpha\cdot\max\{\, d(x_1,x_2),\rho(y_1,y_2),\\
& d(x_1,f_1(x_1,y_1)),d(x_2,f_1(x_2,y_2)),\\
& d(x_1,f_1(x_2,y_2)),d(x_2,f_1(x_1,y_1))\},
\end{aligned}$$
$(x_1,y_1),\ (x_2,y_2)\in X\times Y$;

(ii)
$$\begin{aligned}
\rho(f_2(x_1,y_1),f_2(x_2,y_2)) &\leq \beta\cdot\max\{\, d(x_1,x_2),\rho(y_1,y_2),\\
& \rho(y_1,f_2(x_1,y_1)),\rho(y_2,f_2(x_2,y_2)),\\
& \rho(y_1,f_2(x_2,y_2)),\rho(y_2,f_2(x_1,y_1))\},
\end{aligned}$$
$(x_1,y_1),\ (x_2,y_2)\in X\times Y$;

(iii) $\alpha\in[0;1[$ *şi* $\beta\in[0;1[$ *such that:*

$$\frac{\alpha}{1-\alpha}\cdot\frac{\beta}{1-\beta}<1.$$

In these conditions we have $F_f=\{(x^*,y^*)\}$.

Proof. We'll apply Theorem 2.1 for

$$\begin{aligned}
\psi_1(r_1,r_2,r_3,r_4,r_5,r_6) &= \alpha\cdot\max_{i=\overline{1,6}}\{r_i\},\\
\psi_2(r_1,r_2,r_3,r_4,r_5,r_6) &= \beta\cdot\max_{i=\overline{1,6}}\{r_i\},
\end{aligned}$$

Conditions (iii)-(x) are easy to check.

3 Data dependence theorems

In this section we present a result of data dependence of the fixed points for two operators $f, g : X \times Y \to X \times Y$. For better understanding we give this result in the particular case of Corollary 2.1 when $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ and $\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$, the general case of Theorem 2.1 can be treated similarly..

Theorem 3.1. *Let (X, d), (Y, ρ) be two complete metric spaces and $f, g : X \times Y \to X \times Y$, $f = (f_1, f_2)$, $g = (g_1, g_2)$. Suppose that:*

(i) there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$, with $a_1 < 1$ and $b_2 < 1$, such that

$$d(f_1(x_1, y_1), f_1(x_2, y_2)) \le a_1\, d(x_1, x_2) + a_2 \rho(y_1, y_2),$$

$$\rho(f_2(x_1, y_1), f_2(x_2, y_2)) \le b_1\, d(x_1, x_2) + b_2 \rho(y_1, y_2),$$

$(x_1, y_1),\ (x_2, y_2) \in X \times Y$;

(ii) there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$, with $\alpha_1 < 1$ and $\beta_2 < 1$, such that

$$d(g_1(x_1, y_1), g_1(x_2, y_2)) \le \alpha_1\, d(x_1, x_2) + \alpha_2 \rho(y_1, y_2),$$

$$\rho(g_2(x_1, y_1), g_2(x_2, y_2)) \le \beta_1\, d(x_1, x_2) + \beta_2 \rho(y_1, y_2),$$

$(x_1, y_1),\ (x_2, y_2) \in X \times Y$;

(iii) $\frac{a_2 b_1}{(1-a_1)(1-b_2)} < 1$;

(iv) $\frac{\alpha_2 \beta_1}{(1-\alpha_1)(1-\beta_2)} < 1$;

(v) there exits η_1,$\eta_2 \in \mathbb{R}_+$ such that

$$d\left(f_1(x, y), g_1(x, y)\right) \le \eta_1$$

$$\rho\left(f_2(x, y), g_2(x, y)\right) \le \eta_2$$

for any $(x, y) \in X \times Y$;

Then we have:

$$d\left(x_f^*, x_g^*\right) \leq \eta \cdot \max\left\{\frac{1}{1-\lambda_f}; \frac{1}{1-\lambda_g}\right\},$$

$$\rho\left(y_f^*, y_g^*\right) \leq \tau \cdot \max\left\{\frac{1}{1-\lambda_f}; \frac{1}{1-\lambda_g}\right\},$$

where

$$\lambda_f = \frac{a_2 b_1}{(1-a_1)(1-b_2)} \quad \textit{and} \quad \lambda_g = \frac{\alpha_2 \beta_1}{(1-\alpha_1)(1-\beta_2)},$$

$$\eta = \tau_1 + \tau_2 \cdot \min\left\{\frac{a_2}{1-a_1}; \frac{\alpha_2}{1-\alpha_1}\right\},$$

$$\tau = \tau_2 + \tau_1 \cdot \min\left\{\frac{b_1}{1-b_2}; \frac{\beta_1}{1-\beta_2}\right\},$$

$$\tau_1 = \eta_1 \cdot \min\left\{\frac{1}{1-a_1}; \frac{1}{1-\alpha_1}\right\},$$

$$\tau_2 = \eta_2 \cdot \min\left\{\frac{1}{1-b_2}; \frac{1}{1-\beta_2}\right\}.$$

Proof. Conditions (i)-(iv) show that f, g are in the conditions of Corollary 2.1, thus $F_f = \left\{\left(x_f^*, y_f^*\right)\right\}$ and $F_g = \left\{\left(x_g^*, y_g^*\right)\right\}$. We define the operators P_f, Q_f, respectively P_g, Q_g as in (4) and (5) corresponding to operator f, respectively to operator g. We have that $P_f \circ Q_f$ is λ_f-contraction and $P_g \circ Q_g$ is λ_g-contraction, which mean that $P_f \circ Q_f$ and $P_g \circ Q_g$ are c-Picard operators with constants

$$c_f = \frac{1}{1-\lambda_f} \quad \text{and} \quad c_g = \frac{1}{1-\lambda_g}.$$

We have that

$$d\left(P_f(y), P_g(y)\right) \leq d\left(f_1\left(P_f(y), y\right), g_1\left(P_f(y), y\right)\right) +$$

$$+ d\left(g_1\left(P_f(y), y\right), g_1\left(P_g(y), y\right)\right) \leq \eta_1 + \alpha_1 d\left(P_f(y), P_g(y)\right)$$

and also

$$d\left(P_f\left(y\right),P_g\left(y\right)\right)\leq d\left(g_1\left(P_g\left(y\right),y\right),f_1\left(P_g\left(y\right),y\right)\right)+$$

$$+d\left(f_1\left(P_g\left(y\right),y\right),f_1\left(P_f\left(y\right),y\right)\right)\leq\eta_1+a_1d\left(P_f\left(y\right),P_g\left(y\right)\right)$$

which imply

$$d\left(P_f\left(y\right),P_g\left(y\right)\right)\leq\eta_1\cdot\min\left\{\frac{1}{1-a_1};\frac{1}{1-\alpha_1}\right\}.$$

In a similar way we can prove

$$\rho\left(Q_f\left(x\right),Q_g\left(x\right)\right)\leq\eta_2\cdot\min\left\{\frac{1}{1-b_2};\frac{1}{1-\beta_2}\right\}.$$

We denote by $\tau_1=\eta_1\cdot\min\left\{\frac{1}{1-a_1};\frac{1}{1-\alpha_1}\right\}$ and $\tau_2=\eta_2\cdot\min\left\{\frac{1}{1-b_2};\frac{1}{1-\beta_2}\right\}$.

We have the estimation

$$d\left(P_f\circ Q_f\left(x\right),P_g\circ Q_g\left(x\right)\right)\leq d\left(P_f\left(Q_f\left(x\right)\right),P_g\left(Q_f\left(x\right)\right)\right)+$$

$$+d\left(P_g\left(Q_f\left(x\right)\right),P_g\left(Q_g\left(x\right)\right)\right)\leq\tau_1+\frac{\alpha_2}{1-\alpha_1}\cdot\rho\left(Q_f\left(x\right),Q_g\left(x\right)\right)\leq\tau_1+\frac{\alpha_2}{1-\alpha_1}\cdot\tau_2$$

and also

$$d\left(P_f\circ Q_f\left(x\right),P_g\circ Q_g\left(x\right)\right)\leq d\left(P_g\left(Q_g\left(x\right)\right),P_f\left(Q_g\left(x\right)\right),\right)+$$

$$+d\left(P_f\left(Q_g\left(x\right)\right),P_f\left(Q_f\left(x\right)\right)\right)\leq\tau_1+\cdot\rho\left(Q_f\left(x\right),Q_g\left(x\right)\right)\leq\tau_1+\frac{a_2}{1-a_1}\cdot\tau_2$$

which imply

$$d\left(P_f\circ Q_f\left(x\right),P_g\circ Q_g\left(x\right)\right)\leq\tau_1+\tau_2\cdot\min\left\{\frac{a_2}{1-a_1};\frac{\alpha_2}{1-\alpha_1}\right\}:=\eta.$$

From Theorem 1.1 we conclude

$$d\left(x_f^*,x_g^*\right)\leq\eta\cdot\max\left\{\frac{1}{1-\lambda_f};\frac{1}{1-\lambda_g}\right\}.$$

Using the same technique we can prove that

$$\rho\left(Q_f \circ P_f\left(y\right), Q_f \circ P_f\left(y\right)\right) \leq \tau_2 + \tau_1 \cdot \min\left\{\frac{b_1}{1-b_2}; \frac{\beta_1}{1-\beta_2}\right\} := \tau$$

and therefore, from Theorem 1.1, we conclude

$$\rho\left(y_f^*, y_g^*\right) \leq \tau \cdot \max\left\{\frac{1}{1-\lambda_f}; \frac{1}{1-\lambda_g}\right\}.$$

Thus the theorem is proved.

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