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H-Bases and Interpolation

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

The article presents some results concerning H-bases and theirs applications in multivariate interpolation. We derived the space of reduced polynomials with respect to a particular inner product. We made some connections with least interpolation and presented two application of the connection between spaces of reduced polynomials modulo a H-basis and spaces of ideal interpolation.

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1 Introduction

The article is organized in four sections. In section two, we presented the concept of H-bases for an ideal and the reducing process of a polynomial modulo a vector of polynomial and then, modulo a H-basis of an ideal. This reduction process is dependent of the inner product used. We presented two known inner products and reduced spaces of polynomials. The study of the space of reduced polynomial with respect to the inner product defined in (18) gives the main results of this section.

In section three we presented the connection between the spaces of reduced polynomials modulo the ideal $I = ker\Lambda$ and the interpolation space given by the conditions Λ(we consider only the case when $ker\Lambda$ is a polynomial ideal).

In the last section we presented two applications of the results obtained in section three. These applications use the connection between the least interpolation space Π_Θ, defined by C. de Boor and A. Ron, and the space of reduced polynomials with respect to the inner product defined in (15).

2 Reduction process modulo a H-bases for an ideal

The H-bases concept was introduced by F.S. Macauley and it is based only on the homogeneous terms of polynomials. The references [4], [7], [9] can be use for more details related to H-bases and Gröbner basis.

Next, we will use the notations: Π for the space of all polynomials in "d"-variables, Π_k for the space of polynomials of degree less and equal k, and Π_k^0 for the space of homogenous polynomials of degree in "d" variables.

For any $p \in \Pi$, we call leading term of p, and denote it by $p\uparrow$, the unique homogeneous polynomial for which $deg(p - p\uparrow) < deg\ p$.

Definition 2.1. *A set of polynomials $\mathcal{H} = \{h_1, \ldots, h_s\} \subset \Pi \setminus \{0\}$, is a H-basis for the generated ideal $I =< \mathcal{H} >$ if for any $p \in I$, $p \neq 0$, there is* *an unique representation of* p *in terms of* $\mathcal{H}$*:*

$$(1) \qquad p = \sum_{i=1}^{s} h_i g_i, \;\; g_i \in \Pi \;\; \textit{and} \;\; deg(h_i) + deg(g_i) \leq deg(p).$$

The representation given in (1) is named the H-representation of polynomial p*.*

Proposition 2.1. *A finite set of polynomials,* $\mathcal{H} = \{h_1, \ldots, h_s\} \subset \Pi \setminus \{0\}$*, is a H-basis for the ideal* $I =< \mathcal{H} >$ *if and only if*

$$(2) \qquad M(I) = \{p\uparrow \mid p \in I\} =< p\uparrow \mid p \in \mathcal{H} >,$$

with $M(I)$ *the homogeneous ideal generated by* I*.*

In [2], C. de Boor characterizes a H-basis using its connection with the homogeneous ideals I_k^0.

If $\mathcal{H}$ is a H-basis for the ideal I, then, for all $k \in N$

$$I_k = I \cap \Pi_k = \sum_{h \in \mathcal{H}} p_h \cdot h, \text{ with } p_h \in \Pi_{k-deg(h)}$$

and

$$(3) \qquad I_k^0 = \{p\uparrow \mid p \in I\} \cap \Pi_k^0 = \sum_{h \in \mathcal{H}} \tilde{p}_h \cdot h\uparrow, \text{ cu } \tilde{p}_h = p_h\uparrow \in \Pi^0_{k-deg(h)}.$$

Consequently, $f \in I$, if and only if there exist polynomials $p_h \in \Pi^0_{deg(f)-deg(h)}$ such that

$$(4) \qquad f\uparrow = \sum_{h \in \mathcal{H}} p_h \cdot h\uparrow \;\; \text{and more,}$$

$$(5) \qquad \tilde{f} = f - \sum_{h \in \mathcal{H}} p_h \cdot h \in I$$

Condition (5) allows us to construct a H-basis using an inductive construction of the finite sets $\mathcal{H}_k = \mathcal{H} \cap \Pi_k$, such that

$$I_k = \sum_{h \in \mathcal{H}_k} p_h \cdot h, \text{ with } p_h \in \Pi_{k-deg(h)} \tag{6}$$

Therefore, the H-bases can be used to transform a problem from the infinite space of all polynomials in d variables, in one ore more problems in the finite dimensional spaces of polynomials $I \cap \Pi_k$.

Obviously, any ideal has a H-basis.

The H-bases are deeply connected to the reduction process of a polynomial.

Definition 2.2. *Let be the polynomials $f,\ f_1, \ldots, f_m \in \Pi$. We say that f reduces to $\tilde{f}$ modulo $\mathcal{F} = \{f_1, \ldots, f_m\}$, if the following equality holds*

$$\tilde{f} = f - \sum_{i=1}^{m} g_i \cdot f_i \text{ and } deg(\tilde{f}) < deg(f), \tag{7}$$

and the polynomials g_i satisfy the inequalities:

$$deg(g_i) \leq deg(f) - deg(f_i),\ i = 1, \ldots, m. \tag{8}$$

In that case we use the notation $f \rightarrow_{\mathcal{F}} \tilde{f}$.

We denote by $\stackrel{*}{\rightarrow}_{\mathcal{F}}$ the transitive closure of the binary relation $\rightarrow_{\mathcal{F}}$.

Proposition 2.2. *The finite set $\mathcal{H} = \{h_1, \ldots, h_s\}$ is a H-basis for the ideal $I =< \mathcal{H} >$ if and only if any function $f \in I$ is reduced to 0 modulo $\mathcal{H}$.*

Some finite dimensional vectorial spaces are usefully in order to construct a H-basis for an ideal.

Let $p_1, \ldots, p_m \in \Pi$ and $n \in N$. We defined the spaces:

$$V_n(p_1, \ldots, p_m) = \left\{ \sum_{i=1}^{m} q_i \cdot p_i\uparrow \mid q_i \in \Pi^0_{n-deg(p_i)};\ i = 1, \ldots, m \right\} \subset \Pi^0_n \tag{9}$$

and $\Pi_k^0 = \{0\}$, if $k < 0$.

If I is a polynomial ideal, then we denote by

$$V_n(I) = \{f\uparrow \mid f \in I;\ deg(f) = n\} \subset \Pi_n^0 \tag{10}$$

Let consider an inner product on the polynomial space Π. This inner product induces an orthogonality and, hence we can define the following decomposition in orthogonal components:

$$\Pi_n^0 = V_n(p_1, \dots, p_m) \oplus W_n(p_1, \dots, p_m) \tag{11}$$

$$\Pi_n^0 = V_n(I) \oplus W_n(I) \tag{12}$$

The reduction process given in definition 2.2, can be generalized inductive for every homogeneous components. For a given inner product, a given vector of polynomials $(p_1, \dots, p_m) \in \Pi^m$ and $n \in N$, we make the decomposition of space $V_n(p_1, \dots, p_m)$, defined in (9), in successive orthogonal components:

$$W_n(p_1) = V_n(p_1)$$

$$\dots$$

$$W_n(p_1, \dots, p_j) = V_n(p_1, \dots, p_j) \ominus V_n(p_1, \dots, p_{j-1}),\ j = 2, \dots, m,$$

and we obtain:

$$V_n(p_1, \dots, p_m) = \bigoplus_{j=1}^{m} W_n(p_1, \dots, p_j)$$

In general this decomposition depends on the order of polynomials $p_1, \dots, p_m$.

Proposition 2.3. *(T.Sauer, [9]). For an arbitrary, given order of polynomials in the vector $\mathcal{P} = (p_1, \dots, p_m)$, any polynomial $p \in \Pi$, admits a* *representation in terms of* $\mathcal{P}$*, given by :*

$$p = \sum_{k=1}^{m} q_k \cdot p_k + r, \ \ with\ deg(q_k) + deg(p_k) \leq deg(p)\ \ and \tag{13}$$

$$r = \sum_{n=0}^{deg(p)} r_n, \ \ with\ r_n \perp V_n(p_1, \ldots, p_m). \tag{14}$$

The term r *is named the reduced part of* p *with respect to the vector* $\mathcal{P}$ *and is denoted by* $r = p \rightarrow_{\mathcal{P}}$*.*

The proposition 2.3 is in fact, a Gram -Schmidt type algorithm of orthogonalization and represent a multidimensional generalization of Euclid algorithm (see [7]).

The generalization of the reducing process is given in the following definition:

Definition 2.3. *(T.Sauer, [9]). A polynomial* $f \in \Pi$*, is named reduced with respect to the vector of polynomial* $\mathcal{P} = (p_1, \ldots, p_m)$*, if each homogeneous component of* f *is reduced to zero. Consequently, if*

$$f = \sum_{j=0}^{deg(f)} f_j;\ \ f_j \in \Pi_j^0,\ \ j = 0, \ldots, deg(f)$$

then, we say that f *is reduced with respect to* $\mathcal{P}$ *if and only if*

$$f_j \perp V_j(p_1, \ldots, p_m),\ \ j = 0, \ldots, deg(f).$$

If the reduced part, r, is not zero, it will depend on the order of polynomials in the vector $\mathcal{P}$. The reduced polynomial depends on the inner product used in the direct sum decomposition of the space $V_n(p_1, \ldots, p_m)$. Different inner products, will usually give different classes of reduced polynomials.

H-bases have a special property into reduction process:

Proposition 2.4. *(H. M. Möller, T. Sauer, [7]). Let $\mathcal{H}$ a H-basis for the ideal $I =< \mathcal{H} >$. Then the reduced polynomial r, obtained by the reduction algorithm given in 2.3 is independent of the order of the elements in $\mathcal{H}$.*

The dependence of the spaces of reduced polynomials with respect to a H-basis, using different inner product, was studied by many authors. T. Sauer, in [9], proves the following proposition:

Proposition 2.5. *Let $\mathcal{H} \subset \Pi$ a H-basis for the ideal $I =< \mathcal{H} >$. Then a polynomial q is reduced with respect to the inner product*

$$< f, p >= (p(D)f)(0) = \sum_{\alpha \in N^d} \frac{D^\alpha p(0) D^\alpha f(0)}{\alpha!} \tag{15}$$

if and only if

$$q \in \bigcap_{p \in \mathcal{H}} ker\ p\uparrow(D) = \bigcap_{p \in <\mathcal{H}>} ker\ p\uparrow(D) \tag{16}$$

with $p(D)$ the differential operator with constant coefficients associated to the polynomial p.

W. Gröbner, in [5], proved that, a polynomial is reduced with respect to the inner product

$$(p, q)_* = \sum_{\alpha \in N^d} p_\alpha \cdot q_\alpha, \tag{17}$$

with $p = \sum_{|\alpha| \le grad(p)} p_\alpha x^\alpha$ and $q = \sum_{|\alpha| \le grad(q)} q_\alpha x^\alpha$, if and only if is in Macaulay inverse systems space.

We considered another inner product, we denoted by $< f, g >_a$ and studied the space of reduced polynomials with respect with this inner product.

$$< f, g >_a = \int_0^1 f(at) g(at) dt,\ a \in R^d \setminus \{0\},\ \text{given.} \tag{18}$$

This inner product has an interesting property, that is, only parameter "a" gives to a polynomial the property to be reduced or not with respect to this inner product, whatever is the vector of polynomials with respect to which we make the reduction.

Proposition 2.6. *A polynomial $f \in \Pi_n$ is reduced modulo a vector of polynomials $\mathcal{H} = \{h_1, \ldots, h_m\}$, with respect to the inner product in (18), if and only if $f^{[j]}(a) = 0$, for all $j = 0, \ldots, n$, $f^{[j]}$ being the homogeneous component of f, of degree j.*

Proof. *Necessity:* if f is reduced modulo $\mathcal{H}$, then, using definition 2.3, $f^{[j]} \perp V_j(h_1, \ldots, h_m),$ for all $j = 0, \ldots, n.$ Therefore

$$\int_0^1 f^{[j]}(at)g(at)dt = 0, \text{ for all } g \in V_j(h_1, \ldots, h_m),$$

that is, for any $g = \sum_{i=1}^{m} h_i \uparrow q_i$, with $q_i \in \Pi^0_{j-grad(h_i)}$.

But, both $f^{[j]}$ and g are homogeneous polynomials of degree j. Consequently,

$$\sum_{i=1}^{m} f^{[j]}(a) \cdot (h_i \uparrow q_i)(a) \int_0^1 t^{2j} dt = 0, \text{ for all } q_i \in \Pi^0_{j-deg(h_i)}.$$

Hence $f^{[j]}(a) \cdot \sum_{i=1}^{m} (h_i \uparrow q_i)(a) = 0$, for all $q_i \in \Pi^0_{j-deg(h_i)}$, or equivalent $f^{[j]}(a) = 0$.

Sufficiency: if $f^{[j]}(a) = 0$, for all $j = 0, \ldots, m$, then $\int_0^1 f^{[j]}(at)g(at)dt = 0$, for all $g \in V_j(h_1, \ldots, h_m)$, that is $< f^{[j]}, g >_a = 0$, for all $g \in V_j(h_1, \ldots, h_m)$. Consequently, f is reduced modulo $\mathcal{H}$, with respect to the inner product $< \cdot, \cdot >_a$.

Let denote by $\alpha = (\alpha_1, \ldots, \alpha_d) \in N^d$ a multiindex, $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$. A polynomial is denoted by $f = \sum\limits_{|\alpha| \leq deg\ f} c_\alpha (\cdot)^\alpha$.

Corollary 2.1. *A necessary condition for a polynomial f to be reduced with respect to the inner product $< \cdot, \cdot >_a$, modulo a vector of polynomials is that, for the multiindex α with $|\alpha| = 0$,$c_\alpha = 0$.*

Proposition 2.7. *A polynomial $q \in \Pi$ is reduced with respect to the inner product $< \cdot, \cdot >_a$, modulo an arbitrary vector of polynomials if and only if:*

$$q \in \bigcap_{j=0}^{deg(q)} ker\ \delta_{j,a}, \tag{19}$$

with

$$\begin{aligned} &\delta_{j,a} : \Pi \to R \\ &\delta_{j,a}(f) = f^{[j]}(a), \end{aligned} \tag{20}$$

and $f^{[j]}$ are the homogeneous components of f.

Proof. Let $q \in \Pi_n$ $q \in \bigcap_{j=0}^{n} ker\ \delta_{j,a}$ if and only if $\delta_{j,a}(q) = 0$, for all $j = 0, \ldots, n$ if and only if $q^{[j]}(a) = 0$, for all $j = 0, \ldots, n$, hence proposition 2.7 holds.

3 H-basis and interpolation

Many papers underline the connection between the spaces of reduced polynomials with respect to a H-basis of an zero dimensional ideal , $I = ker\ \Lambda$, ($\Lambda \in \Pi'$, finite) and the interpolation spaces associated to conditions Λ, ([1], [3], [9], [6], etc.).

If Λ gives an ideal interpolation scheme, that is $I = ker\ \Lambda$ is a polynomial ideal, then the ideal I determines the finite dimensional quotient space, $\Pi/_I$, and an equivalence class, with representant f is given by

$$[f] = \{p \in \Pi \mid p - f \in I\}$$

More, we have $dim\ \Pi/_I = card\ \Lambda$.

Two interpolation spaces $\mathcal{P}_1,\ \mathcal{P}_2 \subset \Pi$, with respect to the same conditions Λ, are equivalent modulo I, that is for any $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 - p_2 \in I$ and reciprocally (obviously $dim\ \mathcal{P}_1 = dim\ \mathcal{P}_2 = card\ \Lambda$).

It is proved in [9] and [7], that every H-basis of ideal $I = ker\ \Lambda$ defines a minimal interpolation space.

Theorem 3.1. *Let $\Lambda \subset \Pi'$ be a set of conditions which defines an ideal interpolation scheme and $\mathcal{H}$ a H-basis for the ideal ker Λ. Then the space of polynomials reduced modulo $\mathcal{H}$, $\mathcal{P}_\mathcal{H} = \Pi_{\rightarrow_\mathcal{H}}$, is a minimal interpolation space with respect to Λ, and the interpolation operator associated is the operator of reducing modulo $\mathcal{H}$, that is*

$$L_{\mathcal{P}_\mathcal{H}}(q) = q \rightarrow_\mathcal{H};\ q \in \Pi. \tag{21}$$

Taking into account theorem 3.1 we can construct an ideal of finite codimension, $I = ker\ \Lambda$, $card\ \Lambda < \infty$, if we know a minimal interpolation space for Λ and the interpolation operator.

Proposition 3.1. *(T. Sauer, [6]). Let Λ be a set of conditions which give an ideal interpolation scheme and $I_\Lambda = ker\ \Lambda$. Let L be the interpolation operator of this scheme. Then the ideal I_Λ is given by*

$$I_\Lambda = \{f - L(f) \mid f \in \Pi\} \tag{22}$$

4 Applications

In this section we apply the results in previous section to least interpolation.

Least interpolation is an interpolation scheme introduced by C. de Boor and A. Roon, first for a set of conditions consisting in evaluation functionals on a set of points in R^d.

For any $f \in \mathcal{A}_0$, we define the least term, $f\!\downarrow = T_j f$, with j the smallest integer for which $T_j f \neq 0$ and $T_j f$ the Taylor polynomial of degree $\leq j$. Let $\Theta \subset R^d$ and the spaces:

$$Exp_\Theta = span\{e_\theta;\ \theta \in \Theta\}, \qquad \Pi_\Theta = (Exp_\Theta)\!\downarrow = span\{g\!\downarrow;\ g \in Exp_\Theta\}$$

C. de Boor and A. Roon proved that the pair (Θ, Π_Θ) is always correct. They named this interpolation scheme, "least interpolation".

The main results in this section are two applications given in the propositions 4.1 and 4.

Proposition 4.1. *Let $\Theta = \{\theta_i \mid \theta_i \in R^2,\ i = 1, \ldots, n\}$ a finite set of points situated on unit circle. If $n = 2(q+1)(4q+3)$, $q \in N^*$, then*

$$(23) \qquad \sum_{l=0}^{q+1} \begin{pmatrix} 2q+1 \\ l \end{pmatrix} D^{(2(2q+1-l),2l)} u = 0, \ \textit{for all } u \in \Pi_\Theta,$$

Proof. We know that $I_\Theta = ker\ \Theta$ is a polynomial ideal. Let be $\mathcal{H}$ a H-basis for the ideal I_Θ, with respect to the inner product defined in (15). We observe that $dim\ \Pi_{2(2q+1)} = card\ \Theta$, but

$$(24) \qquad p(x,y) = 1 - (x^2+y^2)^{2q+1} \in \Pi_{2(2q+1)} \cap ker\ \Theta;\ (x,y) \in \mathbb{R}^2$$

Hence, the pair $(\Theta, \Pi_{2(2q+1)})$ is not correct. It is proved by C. de Boor and A. Roon that the pair (Θ, Π_Θ) is minimal correct. Obviously $\Pi_\Theta \neq \Pi_{2(2q+1)}$,

$\Pi_\Theta \subset \Pi_n$, with $n > 2(2q+1)$).

Let be $u \in \Pi_\Theta$. Taking into account theorem 3.1, we obtain that $u \in \Pi_{\to_\mathcal{H}}$ and using proposition 2.5 we obtain

$$u \in \bigcap_{q \in \mathcal{H}} ker\ q\uparrow (D) = \bigcap_{q \in ker\ \Theta} ker\ q\uparrow (D)$$

The polynomial p, defined in (24) satisfies:

$$\begin{aligned} & p \in ker\ \Theta \\ & p\uparrow (D) = \sum_{l=0}^{q+1} \begin{pmatrix} 2q+1 \\ l \end{pmatrix} D^{(2(2q+1-l),2l)}, \end{aligned}$$

therefore $p\uparrow (D)(u) = 0$, for all $u \in \Pi_\Theta$ and that proves (23).

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a set of linear functionals, linear independent. We define the following spaces:

$$H_\Lambda = span\{\lambda^\nu;\ \lambda \in \Lambda\};\ H_\Lambda\downarrow = span\{g\downarrow; g \in H_\Lambda\}, \tag{25}$$

where λ^ν is the generating function of the functional $\lambda \in \Lambda$. This is the general case of "least interpolation". It is known that

Theorem 4.1. *The polynomial subspace $H_\Lambda\downarrow$ is an interpolation space for the set of conditions Λ.*

Proposition 4.2. *([9]) . If Λ gives an ideal interpolation scheme and $\mathcal{H}$ is a H-basis for $I = ker\ \Lambda$ and we consider the reduction process with respect to the inner product given in (15), then*

$$H_\Lambda\downarrow = \Pi_{\to_\mathcal{H}} \tag{26}$$

Let $\Theta = \{\theta_i \mid \theta_i \in \mathbb{R}^2,\ i = 1, \ldots, n\}$ and $\Lambda_{M,N}$ the set of functionals

$$\Lambda_{M,N} = \{\lambda_{m_i,n_i} \mid m_i = \theta_i + a;\ n_i = \theta_i + b;\ a,\ b \in \mathbb{R}^2;\ i = 1, \ldots, n\}$$
$$\lambda_{m_i,n_i}(p) = \int_0^1 p(m_i + (n_i - m_i)t)dt$$

and $\mathcal{H}$ a H-basis for the ideal $I_\Theta = ker\ \Theta$. Then, a polynomial $p \in H_{\Lambda_{M,N}}\downarrow$ if and only if it is reduced modulo $\mathcal{H}$, with respect to the inner product given in (15)

Proof. We proved in [10] that $H_{\Lambda_{M,N}}\downarrow = \Pi_\Theta$. Using corollary 4.2, we obtain $\Pi_\Theta = \Pi_{\to\mathcal{H}}$ and the proposition 4 holds.

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