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Probabilities and Lebesgue measure

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

In this paper we give some applications of Lebesgue measure and we will also estimate an integral from probability theory.

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1. Lebesgue measure is frequent used in problems of the probability theory, in physics and other domains. It is sufficiently to recall that in the probability theory, a Borel measurable application is also a random variable defined on a selection space.

The Lebesgue measure (L) is of great importance in applications on $\mathbb{R}^n$ and we know that is invariant with respect to all the travels.

The next theorems justifies our terminology.

Theorem 1. *Let μ be a σ-finite measure on borelian σ-algebra of the $\mathbb{R}^n$ space. If in addition,*

1. $\mu\{x|0 < x_i \leq 1,\ i = \overline{1,n}\} = 1$

2. *$\mu(E) = \mu(E + a)$ for any borelian set E and for any $a \in \mathbb{R}^n$, then μ is a Lebesgue measure (L) on $\mathbb{R}^n$.*

Theorem 2. *Let L be the Lebesgue measure on $\mathbb{R}^n$ and S be a nondegenerate linear transform on $\mathbb{R}^n$. Then*

$$L(S(E)) = |det\ S| \cdot L(E)$$

holds for any borelian set E.

Theorem 3. (The formula of integration by change of variable).

Let $\Omega \subset \mathbb{R}^n$ be an open set and let L be a Lebesgue measure on Ω. Let $T(x) = (y_1(x), ..., y_n(x))'$, $x = (x_1, ..., x_n)' \in \Omega$ be a given homeomorphism $T : \Omega \to \mathbb{R}^n$ with the continuous derivatives $\frac{\partial y_i}{\partial x_j}, i, j = \overline{1, n}$ on Ω and we note with $\tau(T, x) = \left(\left(\frac{\partial y_i}{\partial x_j}\right)\right), i, j = \overline{1, n}$, the nondegenerate Jacobian matrix for all $x \in \Omega$.

Then for any non-negative borelian function f defined on the open set $T(\Omega)$ we have

$$\int\limits_{T\Omega} f(y)dy = \int\limits_{\Omega} f(Tx) \cdot |\tau(T, x)|dx$$

where by dx, dy we mean the integration with respect to the Lebesgue measure.

Using these theorems it's possible to establish many results with applications in the probability theory, in physics and in mechanics a.s.o.

2. Now, we will refer to an integral on $\mathbb{R}^n$ frequently meet in the probability theory and in mathematical statistics.

For any matrix $\sum$ of the type $(n \times n)$ positive defined with real elements and for any column-defined matrix m, we have

$$(1) \quad \int\limits_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}(x-m)'{\sum}^{-1}(x-m)\right\} dx = (\sqrt{2\pi})^n \cdot \left(\det \sum\right)^{\frac{1}{2}}$$

where x' is the transpose of the $x - vector$.

Since the Lebesgue measure is invariant for any travel, the left hand said will be independent of m. For $m = 0$, because the $\sum$ is positive defined we have the representation $\sum = C'^C$ where C is a nondegenerate matrix.

Here we use the transformation $C\ x = y$ and the left-hand side of the (1) equality is to write

$$|det\ C| \cdot \int\limits_{\mathbb{R}^n} \exp\ \left\{ -\frac{1}{2}(y_1^2 + ... + y_n^2) \right\} dy_1 ... dy_n =$$

$$= |det\ C| \cdot \left(\int\limits_{\mathbb{R}}^{\mathbb{R}^n} e^{-\frac{1}{2}y^2} \cdot dy \right)^n = |det\ C| \left(\sqrt{2\pi} \right)^n = |det\ \sum |^{\frac{1}{2}} \cdot (\sqrt{2\pi})^n.$$

3. Another application refers to finding of the Lebesgue measure of a set having the form

$$A = \{x | P_k(Tx) \leq 1\}$$

where $P_k(x) = \sum\limits_{i=1}^{n} x_i^{2k}$, and T is a nondegenerated linear transform of the $\mathbb{R}^n$ in itself.

Evidently if x goes through on A, the point Tx goes thorough the set $B = \{x / P_k(x) \leq 1\}$.

From the Theorem (2) we have

$$L(B) = L(TA) = |det\ T| \cdot L(A). \tag{2}$$

We note $t_n = L(B)$. But the Lebesgue measure on $\mathbb{R}^n$ is the product of the measures of $\mathbb{R}^{n-1}$ and $\mathbb{R}$.

The section of set B in the point $x_n = x$ is the set

$$B_x = \{(x_1, ..., x_{n-1}) | (x_1^{2k} + ... + x_{n-1}^{2k})\}.$$

On the one hand we have $B_x = \phi$ for $|x| > 1$, and on the other hand for $|x| < 1$ we divide both members of the inequality from the expression Bb_x

by $1-x^{2k}$. In view of relation (2) we have:

$$L(B_x) = t_{n-1}(1-x^{2k})^{\frac{n-1}{2k}}$$

and find that:

$$t_n = L(B) = t_{n-1} \cdot \int\limits_{-1}^{1} (1-x^{2k})^{\frac{n-1}{2k}} \cdot dx =$$

$$= 2t_{n-1} \cdot \int\limits_{0}^{1} (1-y)^{\frac{n-1}{2k}} \cdot y^{\frac{1}{2k}-1} \cdot \frac{dy}{2k} = \frac{t_{n-1}}{k} \cdot \frac{\Gamma\left(\frac{n-1}{2k}+1\right)\Gamma\left(\frac{1}{2k}\right)}{\Gamma\left(\frac{n}{2k}+1\right)}$$

From this recursion formula we obtain

$$t_n = \frac{\left[\Gamma\left(\frac{1}{2k}\right)\right]^n}{k^n \cdot \Gamma\left(\frac{n}{2k}+1\right)}$$

Therefore:

$$L\{x|P_k(T_x) \le 1\} = \frac{|det\ T|^{-1}}{k^n} \cdot \frac{\left[\Gamma\left(\frac{1}{2k}\right)\right]^n}{\Gamma\left(\frac{n}{2k}+1\right)}$$

In particular, the volume of the unite spherical in $\mathbb{R}^n$ is

$$L\left\{x| \sum_i \ x_i^2 \le 1\right\} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

4. In the same way there, can be computed the Lebesgue measure of a standard simplex

$$W = \left\{x|x_i \ge 0, i = \overline{1,n}, \sum_{i=1}^{n} x_i \le 1\right\}$$

Then arque as above to obtain

$$L(W) = \frac{1}{n!}.$$

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