General Mathematics Vol. 11, No. 3–4 (2003), 45–52

Some notes on absolute convergence of Fourier series

Amelia Bucur

Dedicated to Professor Gheorghe Micula on his 60^{th} birthday

Abstract

In this paper we study a necessary and sufficient condition of the absolute convergence of a trigonometric Fourier series is established for continuous 2π-periodic functions which in $[-\pi,\pi]$ have a finite number of intervals of convexity, and whose n-th Fourier coefficients are $O\left(\omega\left(\frac{1}{n};f\right)/n\right)$ where $\omega(\delta;f)$ is the continuity modulus of the function f.

2000 Mathematical Subject Classification: 42A28, 42A16

Will use the following definition: a serie $u_0+u_1+u_2+...$ with real terms is said to be absolutely convergent if the series $|u_0|+|u_1|+|u_2|+...$ of the module of its terms is convergent.

Let ω be an arbitrary modulus of continuity, i.e, a nondecreasing function continuous on [0, 1], $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$. Will use the class of all functions f continuous on $[-\pi, \pi]$ for which

$$\omega(\delta; f) = \sup_{|x_1 - x_2| \le \delta} |f(x_1) - f(x_2)| = O(\omega(\delta)), \quad 0 \le \delta \le 1.$$

Let M be the class of all continuous 2π-periodic functions f for which there exists a partitioning of the segment $[-\pi, \pi]$ by the points $-\pi = x_1(f) < ... < x_{m+1}(f) = \pi$ such that f is convex, or convex, or linear, on each segment $[x_k(f), x_{k+1}(f)]$, $k = 1, ..., m$.

The Fourier coefficients of a function f with respect to the trigonometric system will be denoted by $a_n = a_n(f)$, $b_n = b_n(f)$.

Problems parting to the absolute convergence of Fourier series have been studied quite completely ([4], [5], [6], [7]).

This paper deals with one problem of the absolute convergence of trigonometric Fourier series of a function from class M.

The following fact is well known: the Fourier series of any 2π-periodic continuous even function, convex on $[-\pi, \pi]$, converges absolutely (see [7]).

We have obtained the following result:

Theorem 1. *If $f \in M$, then for absolute convergence of the Fourier series of the function f it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \left| f\left(x_k(f) + \frac{1}{n}\right) - f\left(x_k(f) - \frac{1}{n}\right) \right| \frac{1}{n} < \infty, \quad k = 1, ..., n.$$

Proof. Let f_1, f_2, f be continuous 2π-periodic functions defined as follows: $f_1(x) = 0$ for $x \in [-\pi, 0]$, f_1 is convex or concave on a segment $(0, 1)$, linear on $[1, \pi]$; $f_2(-\pi) = 0$, f_2 is linear on $[-\pi, -1]$, f_2 is convex or concave on $(-1, 0]$, $f_2(x) = 0$ for $x \in (0, \pi]$; $f = f_1 + f_2$.

The theorem will be proved by showing that for Fourier series of f to converge it is necessary and sufficient that

$$\sum_{n=1}^{\infty} \left| f\left(\frac{1}{n}\right) - f\left(-\frac{1}{n}\right) \right| \cdot \frac{1}{n} < +\infty.$$

This follows from Wiener's theorem and from the following facts: If the function f is convex or concave on segment $[a, b]$, then f is a lipschitz function on any segment $[c, d]$ entirely lying inside $[a, b]$, and the Fourier series of the functions $f(x)$ and $f(x + c)$ simultaneously converge or diverge absolutely.

The function f_1 is convex on $[0, \pi]$ and continuous, which means that it is absolutely continuous so that one can apply integration by parts and Newton - Leibnitz formulas to obtain $a_n(f) = a_n(f_1) + a_n(f_2)$.

$$a_n(f_1) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(t) \cos nt \; dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(t) d\frac{\sin nt}{n} =$$

$$= \frac{1}{n} \left(f_1(t) \frac{\sin nt}{t} \Big|_{-\pi}^{\pi} - \frac{1}{\pi n} \cdot \int_{-\pi}^{\pi} f_1'(t) \sin nt \; dt \right) = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f_1'(t) \sin nt \; dt =$$

$$= -\frac{1}{\pi n} \int_{-\pi}^{0} f_1'(t) \sin nt \; dt - \frac{1}{\pi n} \int_{0}^{\pi} f_1'(t) \sin nt \; dt =$$

$$= -\frac{1}{\pi n} \int_{0}^{1/n} f_1'(t) \sin nt \; dt - \frac{1}{\pi n} \int_{1/n}^{1} f_1'(t) \sin nt \; dt - \frac{1}{\pi n} \int_{1}^{\pi} f_1'(t) \sin nt \; dt.$$

The derivative f' of the convex or concave function f is monotonous and therefore, applying the second theorem of the mean value, we obtain:

$$\left|\int_{1/n}^{1} f_1'(t)\sin nt\ dt\right| =$$

$$= \left|\frac{1}{\pi n} f_1'\left(\frac{1}{n}+0\right)\int_{1/n}^{\epsilon}\sin nt\ dt + \frac{1}{\pi n} f_1'(t-0)\int_{\epsilon}^{1}\sin nt\ dt\right| \le$$

$$\le \frac{1}{\pi n^2}\left|f_1'\left(\frac{1}{n}+0\right)\right| + \frac{1}{\pi n^2}\left|f_1'(1-0)\right| \text{ with } \frac{1}{n} < \epsilon < 1.$$

Wherever we come across expressions of the form $f'(x\pm 0)$, the left and right limits are considered with respect to the set at whose points the derivative f' exists.

For the convex (concave) function f we have the relation

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \ge f'(x_2\pm 0) \ge \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

$$\left(\frac{f(x_2)-f(x_1)}{x_2-x_1} \ge f'(x_2\pm 0) \ge \frac{f(x_3)-f(x_2)}{x_3-x_2}\right)$$

where $x_1 < x_2 < x_3$. Therefore

$$\left|f_1'\left(\frac{1}{n}\neq 0\right)\right| \le \frac{f_1\left(\frac{1}{n}\right) - f_1\left(\frac{1}{n+1}\right)}{\frac{1}{n}-\frac{1}{n+1}} \le$$

$$\le (n+1)^2\left(f_1\left(\frac{1}{n}\right) - f_1 - f_1\left(\frac{1}{n+1}\right)\right).$$

Hence

$$\sum_{n=1}^{\infty}\left|f_1'\left(\frac{1}{n}+0\right)\right|\frac{1}{n^2} \le 2\sum_{n=1}^{\infty}\left(f_1\left(\frac{1}{n}\right) - f_1\left(\frac{1}{n+1}\right)\right) < +\infty.$$

Since f_1 is linear on the segment $[\epsilon, \pi]$, we have $|f_1'(1-0)| \leq \beta$ and $\sum\limits_{n=1}^{\infty} |f_1'(1-0)|/n^2 = \sum\limits_{n=1}^{\infty} \beta/n^2 < \infty$.

The function f_1 is linear on the segment $[1, \pi]$, i.e. $f_1'(t) = const\ = \beta$, so that $1/n \left|\int\limits_1^{\pi} f_1'(t) \sin nt\right| \leq \frac{\beta}{n^2}$.

Finally, an $(f_1) = -\frac{1}{n\pi} \int\limits_0^{1/n} f_1'(t) \sin nt\ dt + \gamma_n$, where $\sum\limits_{n=1}^{\infty} |\gamma_n| < +\infty$.

If we introduce the notation $I_n = -\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) \sin nt\ \ dt$, then $a_n(f_1) = I_n + \gamma_n$, $I_n = a_n(f_r) - \gamma_n$.

Since the function f_1 has a bounded variation, we have

$$f_1(x) = \frac{a_0(f_1)}{2} + \sum_{n=1}^{\infty} [a_n(f_1) \cos nx + b_n(f_1) \sin nx].$$

By substituting here $x = 0$ we obtain $\sum\limits_{n=1}^{\infty} a_n(f_1) < \infty$. Therefore $\sum\limits_{n=1}^{\infty} I_n = \sum\limits_{n=1}^{\infty} (a_n(f_1) - \gamma_n) < \infty$.

One can easily verify that the values I_n do not change their sign for sufficiently large n. Thus $\sum\limits_{n=1}^{\infty} (I_n) < +\infty$. Since $|a_n(f_1)| \leq |I_n| + |\gamma_n|$, we obtain $\sum\limits_{n=1}^{\infty} |a_n(f_1)| < +\infty$.

In a similar manner we shall show that $\sum\limits_{n=1}^{\infty} |a_n(f_2)| < \infty$. We have $|a_n(f)| = |a_n(f_1) + a_n(f_2)| \leq |a_n(f_1)| + |a_n(f_2)|$ and $\sum\limits_{n=1}^{\infty} |a_n(f)| < +\infty$.

Now we consider the coefficients $b_n(f)$. We have $b_n(f) = b_n(f_1) + b_n(f_2)$.

$$b_n(f_1) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(t) \sin nt\ dt = \frac{-1}{\pi} \int_{-\pi}^{\pi} f_1(t) d\frac{\cos nt}{n} =$$

$$= -\frac{1}{\pi} f_1(t) \frac{\cos nt}{n} \Big|_{-\pi}^{\pi} + \frac{1}{\pi n} \int_{-\pi}^{\pi} f_1'(t) \cos nt\ dt = \frac{1}{\pi n} \int_{-\pi}^{\pi} f_1'(t) \cos nt\ dt =$$

$$= \frac{1}{\pi n} \int\limits_0^{\pi} f_1'(t) \cos nt \ dt = +\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) \cos nt \ dt + \frac{1}{\pi n} \int\limits_{1/n}^{1} f_1'(t) \cos nt \ dt +$$

$$+\frac{1}{\pi n} \int\limits_1^{\pi} f_1'(t) \cos nt \ dt.$$

The function f_1 is linear on the segment $[1, \pi]$, i.e. $f_1'(t) = const \ = \beta$, so that

$$\frac{1}{n} \left| \int\limits_1^{\pi} f_1' \cos nt \ dt \right| \le \frac{\beta}{n^2}$$

Again applying the theorem of the mean, we obtain (with $\frac{1}{n} < \epsilon < 1$):

$$\left| \frac{1}{n} \int\limits_{1/n}^{1} f_1' \cos nt \ dt \right| =$$

$$= \frac{1}{n} \left| f_1' \left(\frac{1}{n} + 0 \right) \int\limits_{1/n}^{\epsilon} \cos nt \ dt + f_1'(1-0) \int\limits_{\epsilon}^{1} \cos nt \ dt \right| \le$$

$$\le \frac{1}{n^2} \left| f_1' \left(\frac{1}{n} + 0 \right) \right| + \frac{1}{n^2} \left| f_1'(1-0) \right| < +\infty.$$

Therefore $b_n(f_1) = +\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) \cos nt \ dt + \delta_n$, $\sum\limits_{n=1}^{\infty} |\delta_n| < +\infty$. But,

$$\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) \cos nt \ dt = -\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t)(1 - \cos nt - 1) dt = \frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) dt -$$

$$-\frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t)(1 - \cos nt) dt = \frac{1}{\pi n} f_1 \left(\frac{1}{n} \right) - \frac{1}{\pi n} \int\limits_0^{1/n} f_1'(t) \cdot 2 \sin^2 \frac{nt}{2} dt,$$

$$\left|\frac{1}{\pi n}\int\limits_0^{1/n} f_1' 2\sin^2\frac{nt}{2}dt\right| \leq \frac{2}{\pi n}\int\limits_0^{1/n}|f_1'(t)|\cdot\left|\sin^2\frac{nt}{2}\right| = 2|I_n|.$$

As we have seen, above $\sum\limits_{n=1}^{\infty}|I_n| < \infty$ and therefore

$$b_n(f_1) = \frac{1}{\pi n}f_1\left(\frac{1}{n}\right) - C_n = \frac{1}{\pi n}f\left(\frac{1}{n}\right) - C_n,$$

where $\sum\limits_{n=1}^{\infty}|C_n| < +\infty$.

In a similar manner it will be shown that

$$b_n(f_2) = \frac{1}{\pi n}f\left(-\frac{1}{n}\right) + P_n,$$

where $\sum\limits_{n=1}^{\infty}|P_n| < +\infty$.

Since $b_n(f) = b_n(f_1) + b_n(f_2)$, we have

$$b_n(f) = b_n(f_1) + b_n(f_2) + \frac{1}{\pi n}\left\{f\left(\frac{1}{n}\right) + f\left(-\frac{1}{n}\right)\right\} + \gamma_n', \ \sum_{n=1}^{\infty}|\gamma_n'| < \infty,$$

and the proof is completely.

References

[1] Bucur, A., *On some series of functions*, Italian Journal of Pure and Applied Mathematics (accepted for publication in 7.10.2002).

[2] Karchava, J. T., *On absolute convergence of Fourier series*, Georgian Mathematical Journal, vol. 4, 1997, 333 - 340.

[3] Leladze, D., *On some properties of multiple conjugate trigonometric series*, Georgian mathematical Journal, 1, nr.3, 1994, 287 - 302.

[4] Nikolshy, S. M., *A course of Mathematical Analysis*, vol. 1, 2, Mir Publishers Moscow, 1975, English translation, Mir Publishers, 1981.

[5] Roşculeţ, M., *Serii trigonometrice şi aplicaţii*, Ed. Academiei Române, Bucureşti, 1986 (in Romanian).

[6] Vescan, A., *Sumabilitatea seriilor*, Editura Tehnică Bucureşti, 1965 (in Romanian).

[7] Zygmund, A., *Trigonometric series*, v. 1, Cambridge University Press, 1959; Russian translation: Mir, Moscow, 1965.

Department of Mathematics
"Lucian Blaga" University of Sibiu
Str. Dr. I. Raţiu, nr. 5 - 7
550012 - Sibiu, România
E-mail: *amelia.bucur@ulbsibiu.ro*