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Generic properties for two-dimensional dynamic systems

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Abstract

This paper studies the concept of genericity for two dimensional dynamic systems. The basic model is the differential model, proposed by S. Smale, of constructing generic properties for diffeomorphisms defined on compact varieties . The generic property concept for dynamic systems defined on an open set G, is proposed. The aim of the construction is to test the conditions for which, in two-dimensional case, the structural stability of dynamic systems can be a generic property.

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1 The differential model of S. Smale

S. Smale [4] defined the dynamic systems by a diffeomorphism , $f : M \to M$, with M compact differential bundle. Most of results are also available for dynamic systems defined by a first order ordinary differential equation.

The space of all dynamic systems, $DYN(M)$, were topologized with the uniform convergence norm C^r, $1 \leq r \leq \infty$. Thus, studying the dynamic systems implies studying the orbit

$$O(x) = \{f^n(x), n \in Z\}$$

of f, from a global standpoint, and a natural equivalence relation is the topological conjugation:

$f, g \in DYN(M)$ are *topologic conjugated* if there is a homeomorphism $h : M \to M$ so that $f(h(x)) = h(g(x))$ for any $x \in M$ (i.e. h *sends* the orbits of f on those of g).

The central problem of Smale [4] is finding a dense set U (or at least a Baire set) of DYN(M) so that the elements of U could be described by algebraical and/or numerical discrete invariants. But this problem were restrictive, because the basic aspects related to the dynamic systems could not be unified. So another way were chosen: finding a subsets sequence: $U_i \in DYN(M)$, $U_1 \subset U_2 \subset \ldots \subset U_k \subset DYN(M)$, k not very large, so that U_i is open and U_k is dense in $DYN(M)$. The basic feature of U_i is that, if i increases, U_i contains a substantial larger class of dynamic systems, and, on the other hand, when i decreases, the elements of U_i have a better regularity (stability). So, in Smale's model, U_1 contains the simplest class of nontrivial dynamic systems with a good behavior, and U_k cannot have very good stability properties. Moreover, in each U_i there is a large class of elements which are not in U_{i-1}.

In this differential framework, the first four sets were constructed. The case $k \leq 4$, is not completely solved; also, for $k \geq 4$, the questions are still open [4].

The property of *strong transversallity* , related to the proximity of two orbits in M, is of great interest:

Definition 1. Given $f \in DYN(M)$, and a fixed metric on M, we say that $x \approx y$ if $d\left(f^m(x), f^m(y)\right) \longrightarrow 0$, $m \longrightarrow \infty$.

The equivalence class of x, $W^s(x)$, is named the *stable variety* of x.

Remark 1. The *unstable varieties* for f , are stable varieties for f^{-1}.

Thus, the strong transversallity condition for an element x of M means that $W^s(x)$ and $W^u(x)$ *joins transversally in x.*

Smale defined the generic property as follows:

Definition 2. A *generic property* is a property which is true for any Baire set of DYN(M).

Remark 2. There were conjectured that a necessary and sufficient condition for a dynamic system $f \in DYN(M)$ to be structural stable is $f \in U_2$. On the other hand, there were proved that there exists $f \in U_2, f \notin U_1$, being structural stable. This is because the strong transversallity property failed.

That's why, proving the equality:

$$U_2 = \{structural \quad stable\ systems\}$$

would make stronger the sequence $U_1 \subset U_2 \subset U_3$.

2 The construction of the metric space dyn(g) generic properties

As, for the present aim, only the two-dimensional case is tested, instead to compact bundle an open set $G \subset R^2$ is taken. Consequently, $DYN(G)$ will have a simpler geometric structure.

Let us consider the following dynamic system,with f and g sufficiently regular on $G \subset R^2$:

$$\dot{x} = f(x, y) \tag{1}$$

$$\dot{y} = g(x, y).$$

For any natural k, $k \geq 1$, we consider an asociated space of functions, $\bar{A}_{fgk}$, for any dynamic system (1), with the form:

$$\bar{A}_{fgk} =$$

$$= \left\{ f^*, g^* : G \longrightarrow R^2 \mid f^*, g^* \in C^k(G), \text{exists } W_f, W_g \subset G, W_f, W_g \neq \emptyset, \right.$$

$$\left. \textit{open sets};\ f^*(x) = f(x)\ \forall x \in W_f,\ g^*(x) = g(x)\, \forall x \in W_g \right\}$$

Remark 3. A space of functions like $\bar{A}_{fgk}$ can be obviously constructed. It suffices to take the case of real polynomials (of one or fewer real variables), since, for a given polynomial, there always exists at least an approximating polynomial. Moreover, the functions f and g have themselves smoothness conditions.

Let us consider an *unity algebra* structure on $\bar{A}_{fgk}$ (the unity being the identity function 1_G). Thus, we have a function algebra associated to each dynamic system (1), on G.

In this framework, let us construct the sets $U_k, k \geq 1$, of $DYN(G)$:

Definition 3. The dynamic system (1) belongs to U_k if and only if :

a) $f, g \in C^k(G)$;

b) the associated unity algebra $\bar{A}_{fgk}$,is a finite algebra.

Remark 4. a) k is not necessary large; if f and g are analytical functions, k can be as large as possible;

b) in agreement with the definition of the C^k function, or $k_1 < k_2$, any C^{k_2} function will be a C^{k_1} function, thus obtaining the sequence

$$U_1 \supset U_2 \supset U_3 \supset \ldots \supset U_k. \tag{2}$$

In (2) it can be seen that each set U_k has, unlike U_{k-1}, both a larger differentiability order and a richer function algebra $\bar{A}_{fgk}$. Therefore, like in Smale's model, there exists at least a dynamic system belonging to U_{k-1} and not to U_k. Moreover, as k increases, the elements of U_k have a better smoothness, so U_k will contain less, but smoother elements, and the elements of U_1 can be not very smooth.

Let us construct the space $DYN(G)$, considering $G \subset R^2$ an open set and

$$DYN(G) := \bigcup_{k \geq 1} U_k. \tag{3}$$

Remark 5. $DYN(G)$, like is defined above, is finite.

For each $k \geq 1$, let us note e_i the elements of U_k and define the following equivalence relation on U_k:

$$e_1 \approx e_2 \Leftrightarrow \text{exists } h : G \to G \textit{ topologic map} : e_1 h = h e_2 \tag{4}$$

$$(h\, transforms\ the\ orbits\ of\ e_1\ into\ the\ orbits\ of\ e_2)\,.$$

The above equivalence relation is also a topological conjugation relationship, but in our case h is a topological map, not a homeomorphism. Thus, the equivalence class of a dynamic system contain all the dynamic systems which are topologic conjugated with the given system.

Let us consider $k \geq 1$, and $e_1, e_2 \in U_k$:

$(e_1)\, \dot{x} = P_1\,(x, y)\,,\ \dot{y} = Q_1\,(x, y)$

$(e_2)\ \dot{x} = P_2(x,y), \quad \dot{y} = Q_2(x,y),$

and the following distances or the functions P_1 and P_2 :

$$\max_{x,y\in G} = |P_1(x,y) - P_2(x,y)| \tag{5}$$

$$\max_{x,y\in G} \left| P^{(s+l)}_{1x^s y^l}(x,y) - P^{(s+l)}_{2x^s y^l}(x,y) \right|, \ s+l = 1,2,...,r \tag{6}$$

and the corresponding relations for Q_1 and Q_2 :

$$\max_{x,y\in G} = |Q_1(x,y) - Q_2(x,y)| \tag{7}$$

$$\max_{x,y\in G} \left| Q^{(s+l)}_{1x^s y^l}(x,y) - Q^{(s+l)}_{2x^s y^l}(x,y) \right|, \ s+l = 1,2,...,r. \tag{8}$$

Definition 4. The greatest of the numbers (5)-(8) is named the *metric distance* between e_1,and e_2 in U_k. It is denoted d_r.

Definition 5. The metric distance defined below:

$$d = \max_{r=1,...k} d_r \tag{9}$$

is named the *metric distance* on $DYN(G)$.

In this context, it is immediate the following:

Theorem 1. The space $(DYN(G), d)$ is a metric space.

The proof is contained in the above framework.

Remark 6. It is obviously that each of the numbers (5)-(8) is considered between systems of different equivalence classes.

Let us consider a dynamic system like (1). The generic property is defined for (1) like follows:

Definition 6. A certain property of the system (1) is a *generic property* if this property holds in any open set of the metric space $(DYN(G), d)$.

Remark 7. The construction of the metric space $(DYN(G), d)$ using the sequence (2) makes easier the study of the generic properties. Thus, the generic property has the meaning of a hereditary property, standing from U_1 to U_k. Also, the property of a dynamic system has here an analytical meaning.

3 Concluding remarks

1. The function algebra $\bar{A}_{fgk}$ is constructed in order to facilitate the approximate calculus. The construction deals with classical function approximation, no matter to the biological/physical parameter of the dynamic system. Moreover, in $\bar{A}_{fgk}$ there is a *local* approximate calculus.

2. Defining the stable bundle $W^s(x)$, associated to an element of $DYN(G)$, is an open problem.

3. An important feature of the geometric structure of $(DYN(G), d)$ must be outlined: it should be useful to verify if the structural stability is itself a generic property. It is taken into account the s*tructural stability in Andronov's sense* , that implies a *local* study of the structural stability of the trajectories [1]. Taking k sufficient large for the structural stability to be a generic property of the elements of $DYN(G)$, is an open problem.

Working in two-dimensional case is not restrictive. A recent technique, the center manifold method, allows to reduce the order of a dynamic system from m+n equations to n equations, *without influencing the essential analytical features*. Details can be seen in [2].

4. The structure of the metric space $(DYN(G), d)$ can be compared to

a pyramidalstructure. Therefore, as k increase, an improvement of smoothness properties of the elements of $DYN(G)$, on one hand, and a better accuracy, on the other hand, is asked, since at each larger k, U_k contains fewer elements. A model in this sense can be constructed starting from the activator-inhibitor mechanism of Gierer-Meinhardt [3].

References

[1] Androv, A. A.; Leontovich, E. A.; Gordon, I. I.; Maier, A. G., *Theory of bifurcations of dynamic systems on a plane*, John Wiley &Sons, 1973.

[2] Carr, J., *Applications of center manifold theory*, Springer-Verlag, 1980.

[3] Gierer, A.; Meinhardt, H., *A theory of biological pattern formation*, Kybernetik, 12, 1972, 30-39.

[4] Smale, S. *Stability and genericity in dynamical systems*, Lect. Notes in Math., vol. 179, Springer-Verlag, 1970.

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