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Common fixed points and invariant approximations of pointwise R-subweakly commuting maps on nonconvex sets [1]

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Abstract

In this paper we prove a theorem giving sufficient conditions for the existence of common fixed points of pointwise R-subweakly commuting mappings on nonconvex sets. We apply this theorem to derive some results on the existence of common fixed points from the set of best approximation for this class of maps in the set up of metric spaces. The results proved in the paper generalize and extend some known results of F. Akbar and N. Sultana [Anal. Theory Appl. 24(2008) 40-49], W.G. Dotson [J. London Math. Soc. 4(1972) 408-410], N. Hussain [Anal. Theory Appl. 22(2006) 72-80; Demonstratio Math. 39(2006) 389-400], A.R. Khan and A. Latif [Tamkang J. Math. 36(2005) 33-38], D. O'Regan and N. Hussain [Acta Math. Sinica 8(2007) 1505-1508], N. Shahzad [Fixed Point Theory Appl. 1(2005) 79-86] and of few others.

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1 Introduction and Preliminaries

W.G. Dotson Jr. [5] proved some results concerning the existence of fixed points of nonexpansive mappings on a certain class of nonconvex sets. For proving these results, which extended his previous work [4] on starshaped sets, he introduced the following class of nonconvex sets:

Suppose S is a subset of a Banach space E, and let $\mathfrak{F} = \{f_\alpha\}_{\alpha \in S}$be a family of functions from $[0, 1]$ into S, having the property that for each $\alpha \in S$ we have $f_\alpha(1) = \alpha$. Such a family $\mathfrak{F}$ is said to be *contractive* provided there exists a function $\Phi : (0, 1) \to (0, 1)$ such that for all α and β in S and for all t in $(0, 1)$ we have

$$\|f_\alpha(t) - f_\beta(t)\| \le \Phi(t) \, \|\alpha - \beta\|$$

Such a family $\Im$ is said to be *jointly continuous* provided that if $t \to t_\circ$ in $[0, 1]$ and $\alpha \to \alpha_\circ$ in S then $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in S.

This notion can easily be extended to metric spaces. Also it is easy to observe that if S is a starshaped (with z as star center) subset of a normed linear space E and$f_z(t) = tx + (1-t)z$, $x \in S$, $t \in [0, 1]$, then $\mathfrak{F} = \{f_x : x \in S\}$ is a contractive jointly continuous family with $\Phi(t) = t$. Thus the class of subsets of E with the property of contractive and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

Ever since Dotson's work [5], efforts have been made by many researchers (see e.g. [9]-[13]) to extend results proved on convex sets and starshaped sets to the above class of nonconvex sets. F. Akbar et al. [3] extended the concept of pointwise R-subweakly commuting and pointwise R-subcommuting maps, introduced in [17], [18] for starshaped sets to nonstarshaped sets and proved some common fixed point theorems and invariant approximation results for such mappings in p-normed spaces and locally convex topological vector spaces. We extend some of the results of [3], [9], [10], [14] and [19] for such mappings in metric spaces.

To start with, we recall some definitions and known facts to be used in the sequel.

For a metric space (X, d), a continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be (s.t.b.) a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [20].

A subset K of a convex metric space (X, d) is s.t.b. a *convex set* [20] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be *p-starshaped* [6] if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$. The set K is called p-starshaped with $p \in K$ if the segment $[p, x] = \{W(x, p, \lambda) : 0 \leq \lambda \leq 1\}$ joining p to x, is contained in K for all $x \in K$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X, d) is said to satisfy *Property (I)* [7] if for all $x, y, q \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [6], [20]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset K of a metric space (X, d) and $x \in X$, an element $y \in K$ is s.t.b. a *best approximant* to x or a *best K-approximant* to x if $d(x, y) = d(x, K) \equiv \inf\{d(x, y) : y \in K\}$. The set of all such $y \in K$ is denoted by $P_K(x)$.

For a convex subset K of a convex metric space (X, d), a mapping $g : K \to X$ is s.t.b. *affine* if for all $x, y \in K$, $g(W(x, y, \lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0, 1]$.

g is s.t.b. *affine with respect to* $p \in K$ if $g(W(x, p, \lambda)) = W(gx, gp, \lambda)$ for all $x \in K$ and $\lambda \in [0, 1]$.

Suppose (X, d) is a metric space, M is a nonempty subset of X, and S, T are self mappings of M. The mapping T is s.t.b. *S-continuous* if $Sx_n \to Sx$ implies $Tx_n \to Tx$ whenever x_n is a sequence in M and $x \in M$. T is s.t.b. *demi-compact* if any bounded sequence $\{x_n\}$ of points of M satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence. T is s.t.b. an *S-contraction* on M if there exists a $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$, (*$S$-nonexpansive,* if $d(Tx, Ty) \leq d(Sx, Sy)$)for all $x, y \in M$. A point $x \in M$ is a common fixed (coincidence) point of S and T if $x = Sx = Tx$ ($Sx = Tx$). The set of coincidence points is denoted by $C(S, T)$. The pair (S, T) is s.t.b. *(a) commuting* on M if $STx = TSx$ for all $x \in M$ *(b) compatible* [7] if $\lim d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M *(c) weakly compatible* [8] if they commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$ *(d) R-weakly commuting* [15] on M if there exists $R > 0$ such that $d(TSx, STx) \leq Rd(Tx, Sx)$ for all $x \in M$ *(e) pointwise R-weakly commuting* [16] if given $x \in X$, there exists $R > 0$ such that $d(TSx, STx) \leq Rd(Tx, Sx)$ for all $x \in M$.

Suppose (X, d) is a convex metric space, M is a q-starshaped subset of X with $q \in F(S) \cap M$ and is both T- and S- invariant. Then T and S are called *(f) R-subcommuting* [18] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq (R/\lambda)dist(Sx, W(Tx, q, \lambda))$, $\lambda \in [0, 1)$ *(g) R-subweakly commuting* [17] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq Rdist(Sx, W(Tx, q, \lambda))$, $\lambda \in [0, 1)$.

Suppose (X, d) is a metric space, M is a subset of X having a family $\mathfrak{F} = \{f_\alpha\}_{\alpha \in M}$ and is both T- and S-invariant. The pair (T, S) is called *(h) pointwise R-subweakly commuting* on M if for given $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq Rdist(Sx, Y_q^{Tx})$, $Y_q^{Tx} = \{f_{Tx}(k) : 0 \leq k \leq 1\}$ *(i) pointwise R-subcommuting* on M if for given $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq \frac{R}{k} dist(Sx, f_{Tx}(k))$ for each $k \in (0, 1]$ *(j) R-subweakly commuting* on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq Rdist(Sx, Y_q^{Tx})$ *(k) R-subcommuting* on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq \frac{R}{k} dist(Sx, f_{Tx}(k))$ for each $k \in (0, 1]$.

The definition implies that pointwise R-weakly commuting maps commute at their coincidence points. The converse is also true. Thus pointwise R-weak commutativity of S and T at their coincidence points is equivalent to weak compatibility of S and T (see [14], [16]). Compatible maps are weakly compatible but the converse need not be true (see [8]). Compatible maps are obviously pointwise R-weakly commuting but not conversely (see e.g. [16]). It is well known that commuting maps are R-subweakly commuting maps and R-subweakly commuting maps are R-weakly commuting but not conversely (see [17]). R-subcommuting and R-subweakly commuting map are weakly compatible but the converse does not hold (see [17], [18]). Clearly, R-subcommuting, R-subweakly commuting, pointwise R-subcommuting maps are pointwise R-subweakly commuting.

Throughout, we shall write $F(S)$ for set of fixed points of a mapping S

and $F(T,S)$ for set of fixed points of both T and S.

2 Main results

2.1 Common fixed points of pointwise R-subweakly commuting mappings

In this section we discuss the existence of common fixed points of pointwise R-subweakly commuting mappings defined on a subset (not necessarily star-shaped) having contractive jointly continuous family introduced by Dotson [5]. The proved results generalize and extend some of the results of Akbar and Sultana [3], Dotson [4], Hussain [9],[10], Khan et al. [12] and of few others.

We need the following lemma to prove our first theorem:

Lemma 1 *[14] Let S be a closed subset of a metric space (X,d), and T, I be pointwise R-weakly commuting self mappings of S. If cl $(T(S)) \subseteq I(S)$, cl$(T(S))$ is complete, T is I-continuous, and I,T satisfy for all $x,y \in S$ and $0 \le k < 1$,*

$$d(Tx,Ty) \le k\ \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)]\},$$

then $S \cap F(T) \cap F(I)$ is singleton.

Theorem 1 *Let M be a closed subset of a metric space (X,d), and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are pointwise R-subweakly commuting and satisfy for all $x,y \in M$,*

$$\begin{aligned} d(Tx,Ty) \ \le \ & \max\{d(Ix,Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

If T is I-continuous, then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the following conditions holds:

i) cl $(T(M))$ is compact and I is continuous;

ii) M is compact and I is continuous;

iii) M is bounded and I is demicompact.

Proof. For each $n \geq 1$, define $T_n : M \to M$ by $T_n(x) = f_{Tx}(k_n)$, $x \in M$ and $\langle k_n \rangle$ is a sequence in $(0, 1)$ such that $k_n \to 1$. Then T_n is a well defined mapping on M and for each $n \geq 1$, $cl(T_n(M)) \subseteq cl(M) = M = I(M)$. By the contractiveness of the family $\mathfrak{F}$, we get

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{Tx}(k_n), f_{Ty}(k_n)) \\ &\leq \Phi(k_n) d(Tx, Ty). \end{aligned} \tag{1}$$

As T is I-continuous, so each T_n is I-continuous on M and $T_n(x) = f_{Tx}(k_n) \in Y^{Tx}_{f_{Tx}(0)}$ for all $x \in M$ and $n \geq 1$. Thus generalized I-nonexpansiveness of T and inequality (1) imply that

$$\begin{aligned} d(T_n x, T_n y) &\leq \Phi(k_n) \max\{d(Ix, Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ &\quad dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\} \\ &\leq \Phi(k_n)\ \max\{d(Ix, Iy), d(Ix, T_n x), d(Iy, T_n y), d(Ix, T_n y), \\ &\quad d(Iy, T_n x)\} \end{aligned}$$

for all $x, y \in M$. As I and T are pointwise R-subweakly commuting and $I(f_x(k)) = f_{I(x)}(k)$, it follows that for a given $x \in M$,

$$\begin{aligned} d(T_n Ix, IT_n x) &= d(f_{TIx}(k_n), If_{Tx}(k_n)) = d(f_{TIx}(k_n), f_{ITx}(k_n)) \\ &\leq \Phi(k_n) d(TIx, ITx) \leq \Phi(k_n) R dist(Ix, Y^{Tx}_{f_{Tx}(0)}) \\ &\leq \Phi(k_n) R\ d(Ix, T_n x). \end{aligned}$$

This shows that T_n and I are pointwise $\Phi(k_n)R$-weakly commuting for each n.

(i) By Lemma 1, there exists some $x_n \in M$ such that $F(T_n) \cap F(I) = \{x_n\}$ for each n. The compactness of $cl\ (T(M))$ implies that there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y \in M$. Then

$$x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(k_{n_i}) \to f_y(1) = y,$$

and so by the continuity of T and I, we have $y \in F(T) \cap F(I)$. Hence $M \cap F(T) \cap F(I) \neq \emptyset$.

(ii) It follows from (i) as T is continuous.

(iii) By Lemma 1, there exists $x_n \in M$ such that $F(T_n) \cap F(I) = \{x_n\}$ for each n. Since x_n is bounded, $d(x_n, Ix_n) \to 0$, so by the demicompactness of I, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging strongly to $y \in M$. As T is continuous, $Tx_{n_i} \to Ty$. Also $x_{n_i} = T_{n_i}x_{n_i} = f_{Tx_{n_i}}(k_{n_i}) \to f_{Ty}(1) = Ty$. By the uniqueness of the limit, we get $y = Ty$. The result now follows as in (i).

Corollary 1 *Let M be a closed subset of a p-normed space X, and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are pointwise R-subweakly commuting and satisfy for all $x, y \in M$,*

$$\begin{aligned} \|Tx - Ty\|_p \;\leq\; & \max\{\|Ix - Iy\|_p, dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

If T is I-continuous, then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 2 *Let M be a closed subset of a metric space (X, d), and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are pointwise R-subcommuting and satisfy for all $x, y \in M$,*

$$\begin{aligned} d(Tx, Ty) \;\leq\; & \max\{d(Ix, Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

If T is I-continuous, then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 3 *([3]-Theorem 2.2) Let M be a closed subset of a p-normed space X, and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are pointwise R-subcommuting and satisfy for all $x, y \in M$,*

$$\begin{aligned} \|Tx - Ty\|_p \;\leq\; & \max\{\|Ix - Iy\|_p, dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

If T is I-continuous, then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 4 *Let M be a closed subset of a metric space (X,d), and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are R-subweakly commuting and satisfy for all $x,y \in M$*

$$\begin{aligned} d(Tx,Ty) \;&\leq\; \max\{d(Ix,Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ &\tfrac{1}{2}[dist(Ix, Y^{Ty}_{f_{Ty}(0)}) + dist(Iy, Y^{Tx}_{f_{Tx}(0)})]\}. \end{aligned}$$

If T is I-continuous, then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 5 *Let M be a closed subset of a metric space (X,d), and T,I be self mappings of M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are R-subcommuting and satisfy for all $x,y \in M$*

$$\begin{aligned} d(Tx,Ty) \;&\leq\; \max\{d(Ix,Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ &\tfrac{1}{2}[dist(Ix, Y^{Ty}_{f_{Ty}(0)}) + dist(Iy, Y^{Tx}_{f_{Tx}(0)})]\}. \end{aligned}$$

If T is I-continuous, then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 6 *(see [9]-Theorem 3.2) Let M be a closed subset of a p-normed space X, and T, I be self mapping on M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T,I are R-subcommuting and satisfy for all $x,y \in M$*

$$\begin{aligned} \|Tx - Ty\|_p \;&\leq\; \max\{d(\|Ix - Iy\|_p, dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ &\tfrac{1}{2}[dist(Ix, Y^{Ty}_{f_{Ty}(0)}) + dist(Iy, Y^{Tx}_{f_{Tx}(0)})]\}. \end{aligned}$$

If T is I-continuous, then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.

Corollary 7 *Let M be a closed subset of a metric space (X,d), and T,I be self mapping on M such that $M = I(M)$. Suppose that M has a contractive* *jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T is I-nonexpansive. If T and I are R-subweakly commuting, then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.*

Corollary 8 *(see [10]-Theorem 3.2) Let M be a closed subset of a p-normed space X, and T,I be self mapping on M such that $M = I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, and T is I-nonexpansive. If T and I are R-subweakly commuting, then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions (i)-(iii) in Theorem 1 holds.*

Corollary 9 *Let M be a closed subset of a metric space (X,d), and T and I are self mappings of M such that cl $(T(M)) \subseteq I(M)$ and cl $(T(M))$ is compact. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in [0,1)$, I is continuous, and T is I-continuous. If the pair (T,I) is pointwise R-subweakly commuting and satisfies for all $x, y \in M$*

$$\begin{aligned} d(Tx,Ty) \leq \; & \max\{d(Ix,Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Ix, Y^{Ty}_{f_{Ty}(0)}), dist(Iy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

then $M \cap F(T) \cap F(I) \neq \emptyset$.

Corollary 10 *([12]-Theorem 2.3) Let M be a closed q-starshaped subset a normed linear space X, and T and I are self mappings of M such that cl $(T(M))$ $\subseteq I(M)$. Suppose that I is continuous and affine with $q \in F(I)$ and T is I-continuous. If cl$(T(M))$ is compact and the pair (T,I) is pointwise R-subweakly commuting and satisfies for all $x, y \in M$*

$$\begin{aligned} \|Tx - Ty\| \leq \; & \max\{\|Ix - Iy\|, dist(Ix, [Tx,q]), dist(Iy, [Ty,q]), \\ & dist(Ix, [Ty,q]), dist(Iy, [Tx,q])\}, \end{aligned}$$

then $M \cap F(T) \cap F(I) \neq \emptyset$.

Corollary 11 *Let M be a closed subset of a metric space (X,d), and T, I are continuous self mappings of M such that $T(M) \subset I(M)$. Suppose that M has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that*

$I(f_x(k)) = f_{I(x)}(k)$ for all $x \in M$ and $k \in (0,1]$, and cl $(T(M))$ is compact. If T and I are R-subweakly commuting and satisfy for all $x, y \in M$

$$d(Tx, Ty) \leq Q(x, y),$$

where

$$\begin{aligned} Q(x,y) &= \max\{d(Ix,Iy), dist(Ix, Y^{Tx}_{f_{Tx}(0)}), dist(Iy, Y^{Ty}_{f_{Ty}(0)}), \\ & \tfrac{1}{2}[dist(Ix, Y^{Ty}_{f_{Ty}(0)}) + dist(Iy, Y^{Tx}_{f_{Tx}(0)})]\}, \end{aligned}$$

then $F(T) \cap F(I) \neq \emptyset$.

Corollary 12 *([19]-Theorem 2.2) Let M be a closed q-starshaped subset of a p-normed space X, and T, I are continuous self mappings of M such that $T(M) \subset I(M)$. Suppose that I is affine with $q \in F(I)$ and cl $(T(M))$ is compact. If T and I are R-subweakly commuting and satisfy for all $x, y \in M$*

$$\begin{aligned} \|Tx - Ty\| &\leq \max\{\|Ix - Iy\|, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ & \tfrac{1}{2}[dist(Ix, [Ty, q]), dist(Iy, [Tx, q])]\}, \end{aligned}$$

then $F(T) \cap F(I) \neq \emptyset$.

Taking I to be identity map, we have the following result.

Corollary 13 *Let M be a nonempty closed subset of a metric space (X, d), and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$. If T is nonexpansive self map of M and cl $(T(M))$ is compact, then $F(T) \neq \emptyset$.*

Remark 1 *Taking I to be identity map, Theorem 1 extends and generalizes the corresponding results of [4] and [13].*

2.2 Invariant approximation and common fixed points of pointwise R-subweakly commuting mappings

In this section we discuss the existence of common fixed points from the set of best approximation for pointwise R-subweakly commuting mappings.

Theorem 2 *Let T and S be self mappings of a convex metric space (X, d), $u \in F(T, S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$.*

Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,

$$(2) \qquad d(Tx, Ty) \leq \begin{cases} d(Sx, Su) & , \ if \ y = u \\ Q(x, y) & , \ if \ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x,y) = \max\{ & d(Sx, Sy), dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \\ & dist(Sx, Y^{Ty}_{f_{Ty}(0)}), dist(Sy, Y^{Tx}_{f_{Tx}(0)})\}. \end{aligned}$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0, 1)$. If the pair (S, T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the following conditions holds:

i) cl $(T(D))$ is compact and S is continuous;

ii) D is compact and S is continuous;

iii) I is a demicompact map.

Proof. Let $x \in D$, then for any $k \in (0, 1]$, we have

$$d(W(u, x, k), u) \leq kd(u, u) + (1 - k)d(x, u) = (1 - k)d(x, u) < dist(u, M).$$

It follows (see Lemma 3.2, [1]) that the line segment $\{W(u, x, k) : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also $Sx \in P_M(u)$, $u \in F(T) \cap F(S)$, and $\{S, T\}$ satisfy (2), we have

$$d(Tx, u) = d(Tx, Tu) \leq d(Sx, Su) = d(Sx, u) \leq dist(u, M).$$

This implies that $Tx \in P_M(u)$. Consequently, $D = P_M(u)$ is T invariant. Since all the conditions of Theorem 1 are satisfied, $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$ under any one of the conditions (i) to (iii).

Corollary 14 *Let T and S be self mappings of a p-normed space X, $u \in F(T, S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Sx - Su\|_p & , \ if \ y = u \\ Q(x, y) & , \ if \ y \in D \end{cases}$$

where

$$Q(x,y) = \max\{\|Sx-Sy\|_p, dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), dist(Sx, Y^{Ty}_{f_{Ty}(0)}), dist(Sy, Y^{Tx}_{f_{Tx}(0)})\}.$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the conditions (i)-(iii) in Theorem 2 holds.

Corollary 15 *([3]-Theorem 2.8) Let T and S be self mappings of a p-normed space X, $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx-Ty\|_p \le \begin{cases} \|Sx-Su\|_p &, \quad if \ \ y=u \\ Q(x,y) &, \quad if \ \ y \in D \end{cases}$$

where

$$Q(x,y) = \max\{\|Sx-Sy\|_p, dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), dist(Sx, Y^{Ty}_{f_{Ty}(0)}), dist(Sy, Y^{Tx}_{f_{Tx}(0)})\}.$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subcommuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the conditions (i)-(iii) in Theorem 2 holds.

Corollary 16 *Let T and S be self mappings of a convex metric space (X,d), $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$d(Tx,Ty) \le \begin{cases} d(Sx,Su) &, \quad if \ \ y=u \\ Q(x,y) &, \quad if \ \ y \in D \end{cases}$$

where

$$Q(x,y) = \max\{d(Sx,Sy), dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \frac{1}{2}[dist(Sx, Y^{Ty}_{f_{Ty}(0)}) + dist(Sy, Y^{Tx}_{f_{Tx}(0)})]\}.$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the conditions (i)-(iii) in Theorem 2 holds.

Corollary 17 *Let T and S be self mappings of a p-normed space X, $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Sx - Su\|_p & , \quad if \;\; y = u \\ Q(x,y) & , \quad if \;\; y \in D \end{cases}$$

where

$$Q(x,y) = \max\{\|Sx - Sy\|_p, dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \frac{1}{2}[dist(Sx, Y^{Ty}_{f_{Ty}(0)}) + dist(Sy, Y^{Tx}_{f_{Tx}(0)})]\}.$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the conditions (i)-(iii) in Theorem 2 holds.

Corollary 18 *([9]-Theorem 3.4) Let T and S be self mappings of a p-normed space X, $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u)$ is nonempty, $D = SD$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Sx - Su\|_p & , \quad if \;\; y = u \\ Q(x,y) & , \quad if \;\; y \in D \end{cases}$$

where

$$Q(x,y) = \max\{\|Sx - Sy\|_p, dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \frac{1}{2}[dist(Sx, Y^{Ty}_{f_{Ty}(0)}) + dist(Sy, Y^{Tx}_{f_{Tx}(0)})]\}.$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subcommuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$,provided one of the conditions (i)-(iii) in Theorem 2 holds.

The following result generalizes and extends the corresponding results of Al-Thagafi [2], Hussain [9], [10] and Khan et al. [12] proved for normed linear spaces.

Theorem 3 *Let T and S be self mappings of a convex metric space (X,d), $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)(C_M^S(u) = \{x \in M : Sx \in P_M(u)\})$ is nonempty, $D = SD$, S is nonexpansive on $P_M(u) \cup \{u\}$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$d(Tx,Ty) \leq \begin{cases} d(Sx,Su) & , \quad if \;\; y = u \\ Q(x,y) & , \quad if \;\; y \in D \end{cases}$$

where

$$\begin{aligned} Q(x,y) \;\; = \;\; & \max\{d(Sx,Sy), dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \\ & \tfrac{1}{2}[dist(Sx, Y^{Ty}_{f_{Ty}(0)}) + dist(Sy, Y^{Tx}_{f_{Tx}(0)})]\}. \end{aligned}$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$, provided one of the conditions (i) to (iii) in Theorem 2 holds.

Proof. Let $x \in D$, then proceeding as in Theorem 2, we shall get $Tx \in P_M(u)$. As S is nonexpansive on $D \cup \{u\}$, we obtain

$$d(STx,u) = d(STx,Su) \leq d(Tx,u) < dist(u,M).$$

Thus $STx \in P_M(u)$. This implies that $Tx \in C_M^S(u)$ and hence $Tx \in D$ i.e. D is T-invariant. Since all the conditions of Theorem 1 are satisfied,$P_M(u) \cap F(T) \cap F(S) \neq \emptyset$ if *cl* $(T(D))$ is compact or D is compact.

Corollary 19 *([3]-Theorem 2.10) Let T and S be self mappings on a p-normed space X, $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M \cap M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)$ is nonempty, $D = SD$, S is nonexpansive on $P_M(u) \cup \{u\}$ and S and T satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Sx - Su\|_p & , \ if \ \ y = u \\ Q(x,y) & , \ if \ \ y \in D \end{cases}$$

where

$$\begin{aligned} Q(x,y) &= \max\{\|Sx - Sy\|_p , dist(Sx, Y^{Tx}_{f_{Tx}(0)}), dist(Sy, Y^{Ty}_{f_{Ty}(0)}), \\ &\quad \tfrac{1}{2}[dist(Sx, Y^{Ty}_{f_{Ty}(0)}) + dist(Sy, Y^{Tx}_{f_{Tx}(0)})]\}. \end{aligned}$$

Then D is T-invariant. Suppose that T is S-continuous on D, D is closed and has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$. If the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$, provided one of the conditions (i) to (iii) in Theorem 2 holds.

Corollary 20 *Let T and S be self mappings of a convex metric space (X,d), $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)$ is nonempty, closed, $D = SD$, T is S-nonexpansive on $D \cup \{u\}$ and S is nonexpansive on $P_M(u) \cup \{u\}$. Then D is T-invariant. Further, If D has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in [0,1)$ and the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$, provided one of the conditions (i) to (iii) in Theorem 2 holds.*

Corollary 21 *([10]-Theorem 4.1) Let T and S be self mappings of a p-normed space X, $u \in F(T,S)$ for some $u \in X$ and $M \subset X$ such that $T(\partial M) \subset M$. Suppose that $D = P_M(u) \cap C_M^S(u)$ is nonempty, closed, $D = SD$, T is S-nonexpansive on $D \cup \{u\}$ and S is nonexpansive on $P_M(u) \cup \{u\}$. Then D is T-invariant. Further, If D has a contractive jointly continuous family $\mathfrak{F} = \{f_x : x \in M\}$ such that $S(f_x(k)) = f_{S(x)}(k)$ for all $x \in D$ and $k \in (0,1]$ and the pair (S,T) is pointwise R-subweakly commuting, then $P_M(u) \cap F(T) \cap F(S) \neq \emptyset$ if cl $(T(D))$ is compact or D is compact.*

Remark 2 *All the results of this paper remain valid in the set up of a metrizable locally convex topological vector space (X,d), where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x,y)$, for each $\alpha \in (0,1)$ and $x, y \in X$.*

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