4–planar Mappings of Almost Quaternionic and Almost Antiquaternionic Spaces

Jana Bělohlávková, Josef Mikeš and Olga Pokorná [1]

Abstract

4-planar mappings of almost quaternionic and almost antiquaternionic spaces are defined from a unifying point of view. Starting from these definitions fundamental equations of these mappings are found. The presented paper generalizes the results obtained for almost quaternionic spaces. [2]

4-quasiplanar mappings of almost quaternionic spaces have been studied in [4] and [8]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces, see [3], [6], [7], [9], [11], [12], [13]. Similar problems are being studied on complex manifolds in [2]. Antiquaternionc spaces which have been studied e.g. in [5], [10] have some properties similar to those of quaternionic [1]. This fact can be used in the study of 4-planar mappings. Due to these common properties, several questions concerning 4-planar mappings have been unified.

1. A well-known definition says that an *almost quaternionic* space is a differentiable manifold M_n with almost complex structures $\overset{1}{F}$ and $\overset{2}{F}$ defined on it which satisfy

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{2}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{1}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i + \overset{2}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0, \tag{1}$$

where δ^h_i is the Kronecker symbol, see e.g. [1], [3].

The tensor $\overset{3}{F}{}^h_i \equiv \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha$ defines almost complex structure, too. The relations among the tensors $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ are the following

$$\overset{1}{F}{}^h_i = \overset{2}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha = -\overset{3}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha; \ \overset{2}{F}{}^h_i = \overset{3}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha = -\overset{1}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha; \ \overset{3}{F}{}^h_i = \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha = -\overset{2}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha. \tag{2}$$

Any two of the three structures $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ define an initial *almost quaternionic structure.*

In an analogous way we define the *almost antiquaternionic* space by their almost product structures $\overset{1}{F}$ and $\overset{2}{F}$ [5], [10]:

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = \delta^h_i; \quad \overset{2}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i = \delta^h_i; \quad \overset{1}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i + \overset{2}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0. \tag{1'}$$

But the tensor $\overset{3}{F}{}^h_i \equiv \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha$ defines an almost complex structure. The relations among the tensors $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ are the following ones

$$\overset{1}{F}{}^h_i = -\overset{2}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha = \overset{3}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha; \ \overset{2}{F}{}^h_i = -\overset{3}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha = \overset{1}{F}{}^\alpha_i \overset{3}{F}{}^h_\alpha; \ \overset{3}{F}{}^h_i = -\overset{2}{F}{}^\alpha_i \overset{1}{F}{}^h_\alpha = \overset{1}{F}{}^\alpha_i \overset{2}{F}{}^h_\alpha, \tag{2'}$$

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i.e. $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ defines an *almost antiquaternionic structure.*

The tensors $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ and $\overset{1}{{}^*F}, \overset{2}{{}^*F}, \overset{3}{{}^*F}$ define the same almost (anti)quaternionic structure if

$$\overset{\sigma}{{}^*F} = \sum_{\rho=1}^{3} \alpha_\rho \overset{\rho}{F} \tag{3}$$

where α_ρ are some functions.

Let $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ be a space with affine connection Γ without torsion and with almost (anti)quaternionic structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$.

Definition 1 A curve ℓ: $x^h = x^h(t)$ in A_n is called *4-planar* if the tangent vector $\lambda^h = dx^h/dt$ being parallely transported along this curve, remains in the linear 4-dimensional space generated by the tangent vector λ^h and the vectors $\overset{1}{F}{}^h_\alpha\lambda^\alpha$, $\overset{2}{F}{}^h_\alpha\lambda^\alpha$ and $\overset{3}{F}{}^h_\alpha\lambda^\alpha$, which are conjugated to it.

A curve is 4-planar if and only if the equations

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{\rho}\ \overset{s}{F}{}^h_\alpha\lambda^\alpha \tag{5}$$

hold, where $\overset{0}{F}{}^h_i \equiv \delta^h_i$ is the Kronecker symbol, $\Gamma^h_{\alpha\beta}$ are the components of the affine connection of A_n and $\underset{s}{\rho}$ $(s = \overline{0,3})$ denote functions of the parameter t.

Evidently, a 4-planar curve with respect to the structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ is 4-planar with respect to the structure $(\overset{1}{{}^*F}, \overset{2}{{}^*F}, \overset{3}{{}^*F})$ defined by the formulas **(3)**, too. A 4-planar curve is a geodesic curve if $\rho_1 \equiv \rho_2 \equiv \rho_3 \equiv 0$.

2. Consider two spaces with affine connection without torsion A_n and $\bar{A}_n$ with objects of connection Γ and $\overline{\Gamma}$, respectively. Let an almost (anti)quaternionic structure $(\overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ be defined on $\bar{A}_n$.

Definition 2 A diffeomorphism f: $A_n(\Gamma) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ is called *4-planar mapping*, if it maps any geodesic in A_n to a 4-planar curve in $\bar{A}_n$.

We prove the following

Theorem 1 *A mapping from A_n onto $\bar{A}_n$ is 4-planar if and only if in the common coordinate system x with respect to the mapping, the conditions*

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{a}_{(i} \overset{s}{\bar{F}}{}^h_{j)} \tag{6}$$

hold, where Γ^h_{ij} *and* $\overline{\Gamma}^h_{ij}$ *are objects of the affine connections in* A_n *and* $\bar{A}_n$, $\underset{s}{a}{}_i(x)$, $s = \overline{0,3}$, *are covectors,* $\overset{0}{F}{}^h_i \equiv \delta^h_i$, $(i\,j)$ *denotes a symmetrization without division.*

Proof. Let $f\colon A_n \to \bar{A}_n$ be 4-planar. Then any geodesic $\gamma\colon x^h = x^h(t)$ is characterized by the conditions

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho(t)\lambda^h, \tag{7}$$

where $\lambda^h \equiv dx^h/dt$ and ρ is some function of the parameter t. By the definition of a 4-planar mapping, in the common coordinate system x with respect to the mapping, f maps any geodesic to a 4-planar curve of the space $\bar{A}_n$, which is characterized by the equation

$$\frac{d\lambda^h}{dt} + \overline{\Gamma}^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{\rho}\,(t)\,\overset{s}{F}{}^h_\alpha\lambda^\alpha \tag{8}$$

By subtraction of (7) and (8), we obtain the relation

$$P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{a}\;\overset{s}{F}{}^h_\alpha\lambda^\alpha \tag{9}$$

where $\underset{0}{a} = \bar{\rho}_0 - \rho$; $\underset{s}{a} = \bar{\rho}_s$, $s = 1,2,3$ and

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{10}$$

$P^h_{ij}(x)$ is the *tensor of deformation* of the affine connection.

For 4-planar mappings the relations (9) are satisfied identically in any point $x \in A_n$ for any vector field λ^h. Note that the functions $\underset{s}{a}$, $s = \overline{0,3}$ depend not only on the variable $x \in A_n$ but also on the vector field λ^h.

We multiply the formula (9) by $\lambda^i\;\overset{1}{F}{}^j_\alpha\lambda^\alpha\;\overset{2}{F}{}^k_\beta\lambda^\beta\;\overset{3}{F}{}^l_\gamma\lambda^\gamma$ and then we alternate by the indexes h,i,j,k,l. As a result we get the relation

$$\left(P^{[h}_{\alpha\beta}\;\delta^i_\gamma\;\overset{1}{F}{}^j_\delta\;\overset{2}{F}{}^k_\varepsilon\;\overset{3}{F}{}^{l]}_\eta\right)\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\lambda^\varepsilon\lambda^\eta = 0, \tag{11}$$

where $[h\,\dots\,l]$ denotes an alternation.

Clearly, the expression in the parentheses doesn't depend on the vector λ^h. The relation (11) is satisfied for any vector field λ^h. Using this fact, the relation (11) implies

$$P^{[h}_{(\alpha\beta}\;\delta^i_\gamma\;\overset{1}{F}{}^j_\delta\;\overset{2}{F}{}^k_\varepsilon\;\overset{3}{F}{}^{l]}_{\eta)} = 0. \tag{12}$$

Since $n \geq 8$, in any point x there exist two vectors y^h and z^h such that the vectors

$$y^h,\quad \overset{1}{F}{}^h_\alpha y^\alpha,\quad \overset{2}{F}{}^h_\alpha y^\alpha,\quad \overset{3}{F}{}^h_\alpha y^\alpha,\quad z^h,\quad \overset{1}{F}{}^h_\alpha z^\alpha,\quad \overset{2}{F}{}^h_\alpha z^\alpha,\quad \overset{3}{F}{}^h_\alpha z^\alpha,$$

are linearly independent.

First we contract (12) by $y^\alpha y^\beta y^\gamma y^\delta y^\varepsilon y^\eta$. We obtain

$$y^\alpha y^\beta P^{[h}_{\alpha\beta}\ y^i\ \overset{1}{y}{}^j\ \overset{2}{y}{}^k\ \overset{3}{y}{}^{l]} = 0, \tag{13}$$

where $\overset{s}{y}{}^h \equiv \overset{s}{F}{}^h_\alpha y^\alpha$, $s = \overline{0,3}$.

From the relation (13) it follows that

$$P^h_{\alpha\beta} y^\alpha y^\beta = \sum_{s=0}^{3} \underset{s}{a}\ \overset{s}{y}{}^h, \tag{14}$$

where $\underset{s}{a}$ are some functions.

Contracting (12) by $y^\beta y^\gamma y^\delta y^\varepsilon y^\eta$ we obtain, using (14),

$$P^h_{i\alpha} y^\alpha = \sum_{s=0}^{3} \underset{s}{b}\ \overset{s}{F}{}^h_i + \sum_{s=0}^{3} \underset{s}{c}{}_i\ \overset{s}{y}{}^h, \tag{15}$$

where $\underset{s}{b}$ are some functions and $\underset{s}{c}{}_i$ are covectors.

Finally, contracting (12) by $y^\gamma y^\delta y^\varepsilon y^\eta$ and substituting the relations (14) and (15), we see that

$$P^h_{ij} = \sum_{s=0}^{3} \underset{s}{\rho}{}_{(i}\ \overset{s}{F}{}^h_{j)} + \sum_{s=0}^{3} \underset{s}{d}{}_{ij}\ \overset{s}{y}{}^h, \tag{16}$$

where $\underset{s}{\rho}{}_i$, $\underset{s}{d}{}_{ij}$ are some tensors.

Substituting (16) into (12), we obtain

$$\sum_{s=0}^{3} \underset{s}{d}{}_{(\alpha\beta}\ \ \overset{s}{y}{}^{[h}\ \delta^i_\gamma\ \overset{1}{F}{}^j_\delta\ \overset{2}{F}{}^k_\varepsilon\ \overset{3}{F}{}^{l]}_{\eta)} = 0. \tag{17}$$

Similarly, contracting (17) by $z^\alpha z^\beta z^\gamma z^\delta z^\varepsilon z^\eta$, we get

$$\sum_{s=0}^{3} \underset{s}{d}{}_{\alpha\beta}\ z^\alpha z^\beta\ \overset{s}{y}{}^{[h}\ \overset{0}{z}{}^i\ \overset{1}{z}{}^j\ \overset{2}{z}{}^k\ \overset{3}{z}{}^{l]} = 0, \tag{18}$$

where $\overset{s}{z}{}^h \equiv \overset{s}{F}{}^h_\alpha z^\alpha$, $s = \overline{0,3}$.

From linear independence it follows that

$$\underset{s}{d}{}_{\alpha\beta} z^\alpha z^\beta = 0, \quad s = \overline{0,3}. \tag{19}$$

Contracting (17) by $z^\beta z^\gamma z^\delta z^\varepsilon z^\eta$, and using (19), we have

$$\underset{s}{d}{}_{i\alpha} z^\alpha = 0, \quad s = \overline{0,3},$$

and then contracting (17) by $z^\gamma z^\delta z^\varepsilon z^\eta$, and using the preceding facts, we find

$$\underset{s}{d}_{ij} = 0, \quad s = \overline{0,3}.$$

Then the formulas (16) can be written in the form (6). Clearly, the formula (6) is sufficient, too. The theorem is proved.

3. Further, consider two spaces of affine connection without torsion A_n and $\bar{A}_n$ with objects of connections Γ_n and $\overline{\Gamma}_n$ and almost (anti)quaternionic structures $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ and $(\overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$, respectively. We introduce the following.

Definition 3 A diffeomorphism f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ is called a *4-planar mapping of spaces with almost (anti)quaternionic structures*, if any 4-planar curve in A_n is transformed to a 4-planar curve in $\bar{A}_n$ by f.

The next theorem is valid.

Theorem 2 *A mapping f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ is a 4-planar mapping in the common coordinate system x with respect to the mapping f if and only if the following conditions hold*

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{\psi}_{(i} \overset{s}{F}{}^h_{j)}, \tag{20}$$

$$\overset{\tau}{\bar{F}}{}^h_i(x) = \sum_{s=1}^{3} \alpha^\tau_s \overset{s}{F}{}^h_i, \quad \tau = 1, 2, \tag{21}$$

where $\underset{s}{\psi}_i(x)$, $s = \overline{0,3}$ are some vectors and $\alpha^\tau_s(x)$, $s = 1, 2, 3$ are functions satisfying the conditions

$$\sum_{s=1}^{3} (\alpha^\tau_s)^2 = 1, \quad \tau = 1, 2; \qquad \sum_{s=1}^{3} \left(\alpha^1_s \alpha^2_s\right) = 0. \tag{22}$$

Remark. The formulae (21) and (22) express the fact that $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ and $(\overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ define on A_n and $\bar{A}_n$ the same almost (anti)quaternionic structures.

Proof. Let f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ be a 4-planar mapping. Since any geodesic is a special type of a 4-planar curve, the mapping f maps any geodesic to a 4-planar curve.

Then every geodesic in A_n is mapped to a 4-planar curve in $\bar{A}_n$ by this mapping. It follows by Theorem 1 that the formula (6) is satisfied.

Similarly, a special case of 4-planar curves in A_n are the curves ℓ: $x^h = x^h(t)$, for which the tangent vector $\lambda^h = dx^h/dt$ satisfies the equations

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\, \lambda^\alpha \lambda^\beta = \overset{\tau}{F}{}^h_\alpha \lambda^\alpha, \tag{23}$$

where τ is equal to 1 or 2.

By elimination of $\Gamma^h_{\alpha\beta}$ from formulas (23), using (6) we obtain

$$\frac{d\lambda^h}{dt} + \overline{\Gamma}^h_{\alpha\beta}\,\lambda^\alpha\lambda^\beta = \overset{\tau}{F}{}^h_\alpha\lambda^\alpha + \sum_{s=0}^{3} \underset{s}{\psi}{}_\alpha\lambda^\alpha\ \overset{s}{\overline{F}}{}^h_\beta\lambda^\beta. \tag{24}$$

Since this curve in $\bar{A}_n$ must be 4-planar, the condition

$$\overset{\tau}{\overline{F}}{}^h_\alpha\lambda^\alpha = \sum_{s=0}^{3} \underset{s}{a}\ \overset{s}{\overline{F}}{}^h_\alpha\lambda^\alpha, \quad \tau = 1, 2, \tag{25}$$

where $\underset{s}{a}$, $s = \overline{0,3}$, are some functions of arguments $x \in A_n$ and also of vectors λ^h, must be satisfied as well.

Formulas (25) hold in any point $x \in A_n$ and for all vectors λ^h. Multiplying (24) by $\lambda^i\ \overset{1}{\overline{F}}{}^j_\beta\lambda^\beta\ \overset{2}{\overline{F}}{}^k_\gamma\lambda^\gamma\ \overset{3}{\overline{F}}{}^l_\varepsilon\lambda^\varepsilon$ and then alternating over all indexes we obtain

$$\left(\overset{\tau}{F}{}^{[h}_\alpha\,\delta^i_\beta\ \overset{1}{\overline{F}}{}^j_\gamma\ \overset{2}{\overline{F}}{}^k_\delta\ \overset{3}{\overline{F}}{}^{l]}_\varepsilon\right)\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta\lambda^\varepsilon = 0, \qquad \tau = 1, 2.$$

It follows from here that

$$\overset{\tau}{F}{}^{[h}_{(\alpha}\,\delta^i_\beta\ \overset{1}{\overline{F}}{}^j_\gamma\ \overset{2}{\overline{F}}{}^k_\delta\ \overset{3}{\overline{F}}{}^{l]}_{\varepsilon)} = 0, \qquad \tau = 1, 2. \tag{26}$$

Analysing (26) similarly as in the proof of the preceding theorem, we can prove that

$$\overset{\tau}{F}{}^h_i = \sum_{s=0}^{3} \alpha^\tau_s\ \overset{s}{\overline{F}}{}^h_i, \quad \tau = 1, 2, \tag{27}$$

where α^τ_s are some functions.

As $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ and $(\overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ are quaternionic structures, the α^τ_s satisfy conditions analogous to (4). This fact is expressed by the formulas (21) and (22).

On the other hand, the conditions (20), (21) and (22) are evidently sufficient for f to be a 4-planar mapping.

4. Theorem 2 formulates the fact that a 4-planar mapping preserves almost (anti)quaternionic structures.

Therefore, a priori studying such mappings, we shall further suppose that the almost quaternionic structure is preserved, which is in fact equivalent to the condition

$$\overset{s}{\overline{F}}{}^h_i(x) = \overset{s}{F}{}^h_i(x), \qquad s = 1, 2, 3. \tag{28}$$

On the other hand, Theorem 2 implies that under the conditions (28), 4-planar mappings f: $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F}) \to \bar{A}_n(\overline{\Gamma}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$ are characterized by conditions (20).

By these conditions I.N. Kurbatova [4] defined 4-quasiplanar mappings preserving almost-quaternionic structure.

Finally we will suppose that the space $A_n(\Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ is mapped onto the Riemannian space $\bar{V}_n(\bar{g}, \overset{1}{\bar{F}}, \overset{2}{\bar{F}}, \overset{3}{\bar{F}})$.

Theorem 3 *A mapping $f\colon A_n \to \bar{V}_n(\bar{g})$ is 4-planar if and only if the metric tensor $\bar{g}_{ij}(x)$ satisfies the following equations:*

$$\bar{g}_{ij,k} = \sum_{s=0}^{3} \left(\underset{s}{\psi}_{k}\, \bar{g}_{\alpha(i} \overset{s}{F}{}^{\alpha}_{j)} + \underset{s}{\psi}_{(i}\, \bar{g}_{j)\alpha} \overset{s}{F}{}^{\alpha}_{k} \right) \tag{29}$$

where comma is covariant derivative in A_n and the almost (anti)quaternionic structure is preserved.

Proof follows from the fact that formulas (20) and (29) are equivalent.

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Dept. of Mathematics
TU, 17. Listopadu
708 33 Ostrava, Czech.Rep.
e-mail: Jana.Belohlavkova@vsb.cz

Dep. of Algebra and Geometry
Fac. Sci., Palacky Univ., Tomkova 40
779 00 Olomouc, Czech Rep.
e-mail: Mikes@matnw.upol.cz

Dept. of Mathematics
School of Agriculture, Kamýcká
Praha 6, Czech Rep.
e-mail: Pokorna@tf.czu.cz