Coherent States and Geometry

Ştefan Berceanu

Abstract

The coherent states are viewed as a powerful tool in differential geometry. It is shown that some objects in differential geometry can be expressed using quantities which appear in the construction of the coherent states. The following subjects are discussed via the coherent states: the geodesics, the conjugate locus and the cut locus; the divisors; the Calabi's diastasis and its domain of definition; the Euler-Poincaré characteristic of the manifold, the number of Borel-Morse cells, Kodaira embeding theorem....

1 Introduction

The coherent states were invented by Erwin Schrödinger in the physical context of the quantum harmonic oscillator [1]. Lately, they were used in Quantum Optics by Roy Glauber, E. C. G. Sudarshan and John Klauder [2]. Robert Gilmore [3] and Askold Perelomov [4] have introduced a group theoretical setting which contains Glauber's coherent states as the particular case of the Heisenberg-Weyl group. Perelomov's definition of coherent states was globalized by John Rawnsley [5]. Perelomov's coherent state vectors $\boldsymbol{e}_Z$ are points in a Hilbert space $\mathcal{H}$ parametrized by the points Z of the so called coherent state manifold $\widetilde{\mathbf{M}}$, while Rawnsley's coherent state vectors $\boldsymbol{e}_q$ are indexed by the points q of the so called quantum line bundle $\mathbf{M}$ over $\widetilde{\mathbf{M}}$. In the case of homogeneous Hodge manifolds $\widetilde{\mathbf{M}}$, the quantum line bundle is the pull-back of the hyperplane bundle $[1_N]$ on the projective space $\mathbb{CP}^N$ in which $\widetilde{\mathbf{M}}$ is embedded [6]. However, the transition probability, i.e. the square of the scalar product of the coherent vectors in $\mathcal{H}$, depends in both cases on the points of the manifold $\widetilde{\mathbf{M}}$.

Actually, a huge literatature in Physics and Mathematical Physics is devoted to the coherent states. Here I shall stress the mathematical impact of the coherent states. I shall show that using the coherent states some results in differential geometry can be easily recoverd.

More precisely, in this paper I show how the coherent states permit to find: 1) the geodesics; 2) the conjugate locus; 3) the cut locus; 4) the divisors; 5) the Calabi's diastasis and its domain of definition; 6) the Euler-Poincaré characteristic of the manifold, the number of Borel-Morse cells, Kodaira embeding theorem.... These results are true for very particular manifolds, to be stated precisely in every case. For the moment, I was not able to produce new mathematical information via the coherent state aproach. However, I find very important that the mathematical objects in the list above can be expressed using some quantities which have a physical relevance. Even more, some new remarks have emerged. For the example, the remark **Polar divisor = Cut locus**, proved on naturally reductive spaces, gives a very simple characterization of the cut locus in terms of coherent states. Of course, it will be interesting to see how far these properties can be extended.

The original part of this paper is extracted from the references [7] – [18]. The program of investigation of coherent states developed here was firstly announced in [9]. A short version of this work is contained in [12]. Reference [13] contains an illustration of the methods claimed in this paper on the complex Grassmann manifold $G_n(\mathbb{C}^{m+n})$. I remeber that in [13] I have pointed out some open problems referring to the conjugate locus on $G_n(\mathbb{C}^{m+n})$. Here I present only sketches of the proofs, because some of them have been already published, while the rest are still in preparation ([17, 18]).

2 Preliminaries: the coherent states

Firstly some notation is introduced at 1.). Then the coherent states are defined at 2.).

1.) Let χ be a continuous representation of the group K on the Hilbert space $\mathcal{K}$ and let us consider the principal bundle

$$K \xrightarrow{i} G \xrightarrow{\lambda} \widetilde{\mathbf{M}}, \tag{2.1}$$

where $\widetilde{\mathbf{M}}$ is diffeomorphic with G/K, i is the inclusion and λ is the natural projection $\lambda(g) = gK$. Let $\mathbf{M}_\chi := \widetilde{\mathbf{M}} \times_\chi \mathcal{K}$, or simply $\mathbf{M} := \widetilde{\mathbf{M}} \times_K \mathcal{K}$, be the *G-homogeneous vector bundle* [19] associated by the character χ to the principal K-bundle (2.1). Let $U \subset \widetilde{\mathbf{M}}$ be open. We introduce the notation

$$(G)^U = \{g \in G | \pi(g)\psi_0 \in U\}, \tag{2.2}$$

where π is a representation of G whose restriction to K is χ and $\psi_0 \in \mathcal{K}$ corresponds to the base point $o \in \widetilde{\mathbf{M}}$. Then the continuous sections of $\mathbf{M}_\chi$ over U are precisely the continuous maps $\sigma : U \rightarrow G \times_\chi \mathcal{K}$ of the form

$$\sigma(\pi(g)\psi_0) = [g, e_\sigma(g)], \ e_\sigma : (G)^U \rightarrow \mathcal{K}, \tag{2.3}$$

where e_σ satisfies the "functional equation":

$$e_\sigma(gp) = \chi(p)^{-1} e_\sigma(g), g \in (G)^U, p \in K. \tag{2.4}$$

For homogeneous holomorphic line bundles [20] ($\mathcal{K} = \mathbb{C}$) the functions in eq. (2.4) are holomorphic.

2.) Let $\xi : \mathcal{H}^\star = \mathcal{H} \setminus \{0\} \rightarrow \mathbf{P}(\mathcal{H})$, $\xi(\boldsymbol{z}) = [\boldsymbol{z}]$ be the mapping which associates to the point $\boldsymbol{z}$ in the punctured Hilbert space the linear subspace $[\boldsymbol{z}]$ generated by $\boldsymbol{z}$, where $[\lambda \boldsymbol{z}] = [\boldsymbol{z}], \lambda \in \mathbb{C}^*$. The hermitian scalar product $(\cdot, \cdot)$ on $\mathcal{H}$ is linear in the second argument.

Let us consider the principal bundle (2.1) and let us suppose the existence of a map $e : G \rightarrow \mathcal{H}^*$ as in eq. (2.3) with the property (2.4) but globally defined, i.e. on the neighbourhood (2.2) $(G)^{\widetilde{\mathbf{M}}}$. Then $e(G)$ is called *family of coherent vectors* [5]. If there is a morphism of principal bundles [21], i.e. the following diagram is commutative,

$$
\begin{array}{ccc}
G & \xrightarrow{e} & \mathcal{H}^\star \\
\lambda\downarrow & & \downarrow\xi \\
\widetilde{\mathbf{M}} & \xrightarrow{\iota} & \mathbf{P}(\mathcal{H})
\end{array}
\tag{2.5}
$$

then $\iota(\widetilde{\mathbf{M}})$ is called *family of coherent states corresponding to the family of coherent vectors* $e(G)$ [5].

We restrict ourselves to the case where the mapping ι is an embedding in some projective Hilbert space

$$\iota : \widetilde{\mathbf{M}} \hookrightarrow \mathbf{P}(\mathcal{H}). \tag{2.6}$$

Also we consider only *homogeneous* manifolds $\widetilde{\mathbf{M}}$ [4]. In this paper we shall restrict ourselves to *kählerian embeddings* (cf. Ch. 8 in [22]) ι, i.e.

$$\omega_{\widetilde{\mathbf{M}}} = \iota^*\omega_{\mathbf{P}(\mathcal{H})}, \tag{2.7}$$

where ω is the fundamental two-form (i.e. closed, (strongly) non-degenerate) of the Kähler manifold and ι^* is the pull-back of the mapping ι.

The manifold $\widetilde{\mathbf{M}}$ is called *coherent state manifold* and the G-homogeneous line bundle $\mathbf{M}_\chi$ is called *coherent vector manifold* [7]. We shall also suppose that the line bundle $\mathbf{M}$ is *very ample.*

Let now $\tilde{\pi}$ be a projective (in physical literature [23] "ray") representation associated to the unitary irreducible representation π and $\tilde{G}$ the group of transformations which leaves invariant the transition probabilities in the complex separable Hilbert space $\mathcal{H}$. If we use the projection $\xi' = \xi_{|\mathcal{S}(\mathcal{H})}$, i.e.

$$\xi' : \mathcal{S}(\mathcal{H}) \rightarrow \mathbf{P}(\mathcal{H}),\ \xi'(\psi) = \tilde{\psi} = \{e^{i\phi}\psi | \phi \in \mathbb{R}\},$$

where $\mathcal{S}(\mathcal{H})$ is the unit sphere in $\mathcal{H}$, then $\tilde{\pi}\circ\xi' = \xi'\circ\pi_{|\mathcal{H}}$. The triplet $(\tilde{\pi}, \tilde{G}, \mathcal{H})$ is a quantum system with symmetry in the sense of Wigner and Bargmann [24, 23]. Then the manifold $\widetilde{\mathbf{M}} \approx G/K$ can be realized as the orbit $\widetilde{\mathbf{M}} = \{\tilde{\pi}(g)\tilde{\boldsymbol{e}}_0 | g \in G\}$, where K is the stationary group of $\tilde{\boldsymbol{e}}_0$ and $\boldsymbol{e}_0 \in \mathcal{H}^\star$ is fixed. For a compact connected simply connected Lie group G, the existence of the representation $\tilde{\pi}$ implies the existence of the unitary irreducible representation π (cf. the theorem of Wigner and Bargmann [24, 23]). This implies the existence of cross sections $\sigma : \widetilde{\mathbf{M}} \rightarrow \mathcal{S}(\mathcal{H})$. However, the (Hopf) principal bundle $\boldsymbol{\xi}' = (\mathcal{S}(\mathcal{H}), \xi', \mathbf{P}(\mathcal{H}))$ is a $U(1)$-bundle and in the construction of coherent vector manifold we need line bundles. But the principal line bundle $\boldsymbol{\xi}'$ is obtained from the (tautological) line bundle $[-1] = \boldsymbol{\xi} = (\mathcal{H}^\star, \xi, \mathbf{P}(\mathcal{H}))$ reducing the group structure from $\mathbb{C}^* = GL(1,\mathbb{C})$ to $U(1)$.

Here we also stress that the theorem of Wigner and Bargmann is essentially [25] the (first) fundamental theorem of projective geometry [26].

In order to have the physical interpretation of the "classical system" obtained by dequantizing the quantum one [27, 10], we have restricted ourselves to Kähler manifolds $\widetilde{\mathbf{M}}$. For example, for a compact connected simply connected Lie group G,

$\widetilde{\mathbf{M}} \approx G/K \approx G^{\mathbb{C}}/P$ is a flag manifold [28] and the Borel-Weil theorem [29] assures the geometrical realisation of the representation π_j and of the representation space $\mathcal{H}_j$ if $\boldsymbol{e}_0 = j$. Here P is a parabolic subgroup of the compexification $G^{\mathbb{C}}$ of G and j is the dominant weight.

The representation π_j can be uniquely extended to the group homomorphism $\pi^*_j: G^{\mathbb{C}} \rightarrow \pi^*_j(G^{\mathbb{C}})$, and respectively, Lie algebra isomorphism $\dot{\pi}^*_j$: $\mathfrak{g}^{\mathbb{C}} \rightarrow \dot{\pi}^*_j\ (\mathfrak{g}^{\mathbb{C}})$ by

$$\pi^*_j(\mathrm{e}^Z) = \exp(\dot{\pi}^*_j\ (Z)), Z \in \mathfrak{g}^{\mathbb{C}}, \tag{2.8}$$

where $\mathrm{e} : \mathfrak{g}^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ and $\exp : \dot{\pi}^*_j\ (\mathfrak{g}^{\mathbb{C}}) \rightarrow \pi^*_j(G)$ are exponential maps, while $\dot{\pi}^*_j\ (\mathfrak{g}^{\mathbb{C}})$ is the complexification of the Lie algebra $\dot{\pi}_j\ (\mathfrak{g})$. We use also the notation $F_\alpha = \dot{\pi}^*_j\ (f_\alpha)$, where α is in the set Δ of the roots of the Lie algebra $\mathfrak{g}$ of G with generators f_α of the Cartan-Weyl base of $\mathfrak{g}^{\mathbb{C}}$ (see also [7]).

Then $\boldsymbol{e}_g := e(g) := \pi^*(g)\boldsymbol{e}_0, g \in G^{\mathbb{C}}$ is the family of coherent vectors, while $\{\tilde{\boldsymbol{e}}\}_{g \in G^{\mathbb{C}}}$ is the family of coherent states. The relation $\boldsymbol{e}_g = \mathrm{e}^{i\alpha(g)}\boldsymbol{e}_{\lambda(g)}$ defines a fibre bundle with base $\widetilde{\mathbf{M}}$ and fibre $U(1)$ [4]. More precisely, *the function*

$$\boldsymbol{\Upsilon}(g) = (\Upsilon, \boldsymbol{e}_g) \tag{2.9}$$

is holomorphic on $G^{\mathbb{C}}$ and defines holomorphic sections on the homogeneous holomorphic line bundle $\mathbf{M}$ associated to the principal line bundle $P \rightarrow G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}/P$ by the holomorphic character χ

$$\pi^*(p)e_0 = \chi^{-1}(p)e_0,\ p \in P,\ \chi(p) = \mathrm{e}^{-i\alpha(p)}. \tag{2.10}$$

Indeed, the function $\boldsymbol{\Upsilon}(g)$ verifies $\boldsymbol{\Upsilon}(gp) = \chi^{-1}(p)\boldsymbol{\Upsilon}(g), g \in G^{\mathbb{C}},\ p \in P$, i.e. eq. (2.4), and the corresponding holomorphic sections are associated via eq. (2.3).

Let also the function

$$\boldsymbol{\Upsilon}'(g) = \boldsymbol{\Upsilon}'(gP) := \frac{\boldsymbol{\Upsilon}(g)}{(\boldsymbol{e}_0, \boldsymbol{e}_g)}, \tag{2.11}$$

defined on the set

$$(\boldsymbol{e}_0, \boldsymbol{e}_g) \neq 0. \tag{2.12}$$

Then

$$\boldsymbol{\Upsilon}' : \mathcal{V}_0 \rightarrow \mathbb{C},\ \boldsymbol{\Upsilon}'(Z) = (\Upsilon, \boldsymbol{e}_{Z,j}), \tag{2.13}$$

where the Perelomov's coherent vectors are

$$\boldsymbol{e}_{Z,j} = \exp \sum_{\varphi \in \Delta^+_n} (Z_\varphi F^+_\varphi)j, \quad \underline{\boldsymbol{e}}_{Z,j} = (\boldsymbol{e}_{Z,j}, \boldsymbol{e}_{Z,j})^{-1/2}\boldsymbol{e}_{Z,j}, \tag{2.14}$$

$$\boldsymbol{e}_{B,j} = \exp \sum_{\varphi \in \Delta^+_n} (B_\varphi F^+_\varphi - \bar{B}_\varphi F^-_\varphi)j, \quad \boldsymbol{e}_{B,j} := \underline{\boldsymbol{e}}_{Z,j}. \tag{2.15}$$

Here Δ^+_n denotes the positive non-compact roots, $Z := (Z_\varphi) \in \mathbb{C}^n$ are local coordinates in the maximal neighbourhood $\mathcal{V}_0 \subset \widetilde{\mathbf{M}}$. Also

$$F_{\varphi}^{\pm} = \dot{\pi}^{*}\,(f_{\varphi}^{\pm}),\ \varphi \in \Delta_n^{+}, \tag{2.16}$$

where

$$f_{\varphi}^{\pm} = \begin{cases} k_{\varphi}^{\pm} = ie_{\pm\varphi}, & \text{for } X_n, \\ e_{\varphi}^{\pm} = e_{\pm\varphi}, & \text{for } X_c. \end{cases} \tag{2.17}$$

$e_{\varphi}^{\pm} = e_{\pm\varphi}$ are the part of the Cartan-Weyl base corresponding to $\mathfrak{m}$. Here

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{2.18}$$

is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ and $\mathfrak{k}$ is the Lie algebra of $\mathbf{K}$. The subindex c (n) in eq. (2.17) denotes the compact (respectively, noncompact) manifold $X_{c,n} \approx \widetilde{\mathbf{M}}$. Then $\mathfrak{m}$ is identified with the tangent space at o, $\widetilde{\mathbf{M}}_o$, and $\widetilde{\mathbf{M}} \approx \mathrm{e}^{\mathfrak{m}}$.

The homogeneous symmetric spaces are obtained as

$$X_{n,c} = \mathrm{e}^{\sum_{\varphi\in\Delta_n^{+}}(B_{\varphi}f_{\varphi}^{+} - \bar{B}_{\varphi}f_{\varphi}^{-})}\ \cdot\ o, \tag{2.19}$$

where $o = \lambda(e)$ and e is the unit element in G.

In eqs. (2.14), (2.15)

$$F_{\varphi}^{+}j \neq 0,\ F_{\varphi}^{-}j = 0,\ H_i j = j_i j, \tag{2.20}$$

where $\varphi \in \Delta_n^{+}$, $H_i = \pi^{*}(h_i)$, $\{h_i\}$ is a base of the Cartan subalgebra and $i = 1, \ldots, \operatorname{rank} G$.

The system $\{\boldsymbol{e}_g\}$, $g \in G^{\mathbb{C}}$ is overcomplete [30, 4, 31] and $(\boldsymbol{e}_g, \boldsymbol{e}_{g'})$, up to a factor, is a *reproducing kernel* for the holomorphic vector bundle $\xi_0 : \mathbf{M} \to \widetilde{\mathbf{M}}$ [32].

3 The geodesics

Let us consider again the orthogonal decomposition (2.18) of $\mathfrak{g}$ with respect to the B-form and let also $\mathrm{Exp}_p : \widetilde{\mathbf{M}}_p \to \widetilde{\mathbf{M}}$ be the geodesic exponential map.

Let us consider the following two conditions

A) $\mathrm{Exp}_o = \lambda \circ \mathrm{e}|_{\mathfrak{m}}$.

B) On the Lie algebra $\mathfrak{g}$ of G there exists an $Ad(G)$-invariant, symmetric, non-degenerate bilinear form B such that the restriction of B to the Lie algebra $\mathfrak{k}$ of K is likewise non-degenerate.

We point out that *if the homogeneous space* $\widetilde{\mathbf{M}} \approx G/K$ *verifies* B), *then it also verifies* A) (cf. Corollary 2.5, Thm. 3.5 and Corollary 3.6 Chapter X in Ref. [33]). Indeed, if $\mathfrak{g} = \mathfrak{k} \oplus \mathbf{m}$ is the orthogonal decomposition relative to the B-form on $\mathfrak{g}$, then $\mathfrak{m}$ is canonically identified with the tangent space at o, $\widetilde{\mathbf{M}}_o$. B) implies a (possibly indefinite) G-invariant metric on $\widetilde{\mathbf{M}}$. It follows that G/K is reductive, i.e. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. If B) is true, then $\widetilde{\mathbf{M}}$ is *naturally reductive* (see p. 202 in Ref. [33]) and A) is also verified. The symmetric spaces verify besides the conditions of reductive spaces, the condition $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and, of course, A) is verified too (see Thm. 3.2 Ch. Xl in Ref. [33]).

Thimm [34] furnishes as another examples of homogeneous spaces verifying B), besides the symmetric spaces, the Lie groups with bi-invariant metric and the normal homogeneous spaces (i.e. B is positive definite). Kowalski [35] studied generalised symmetric spaces still verifying condition A). See also the reference [36] for more recent results on naturally reductive spaces and [37].

Now we remember that in Ref. [8] we did the following Remark, which is in fact E. Cartan's theorem (see e.g. Thm. 3.3 p. 208 in Ref. [38]) on geodesics on symmetric spaces expressed in the coherent state setting:

Proposition 1 *The vector* $\boldsymbol{e}_{tB,j} = \exp \dot{\pi}^*_j\ (tB)j \in \mathbf{M}, B \in \mathfrak{m}$, *describes trajectories in* $\mathbf{M}$ *corresponding to the image in the manifold of coherent states* $\widetilde{\mathbf{M}} \hookrightarrow \mathbf{P}(\mathcal{H})$ *of geodesics through the identity coset element on the symmetric space* $X \approx G/K$. *The dependence* $Z(t) = Z(tB)$ *appearing when one passes from eq. (2.15) to eq. (2.14) describes in* $\mathcal{V}_0$ *a geodesic.*

We shall reformulate Proposition 1 in a way very useful even for practical calculations. The proof presented below implies also Thm. 1.

Proposition 2 *For an* $n-$ *dimensional manifold* $X \approx G/K$ *which has Hermitian symmetric space structure, the parameters* B_φ *in formula (2.15) of normalised coherent states are normal coordinates in the normal neighbourhood* $\mathcal{V}_0 \approx \mathbb{C}^n$ *around the point* $Z_\varphi = 0$ *on the manifold* X.

Proof. The Harish-Chandra embedding theorem can be used (cf. e.g. Ref. [39]; see also Ref. [8] for the present context). This theorem asserts that the map $M^+ \times K^{\mathbb{C}} \times M^- \rightarrow G^{\mathbb{C}}$ given by $(m^+, k, m^-) \rightarrow m^+ k m^-$ is a complex analytic diffeomorphism onto an open dense subset of $G^{\mathbb{C}}$ that contains G_n. Let $\mathfrak{m}^\pm$ be the $\pm i$ eigenspaces of the complex structure J and $M^\pm$ the (unipotent, Abelian) subgroups of $G^{\mathbb{C}}$ corresponding to $\mathfrak{m}^\pm$. Then, in particular, $b : \mathfrak{m}^+ \rightarrow X_c = G^{\mathbb{C}}/P,\ b(X) = \mathrm{e}^X P$ is a complex analytic diffeomorphism of $\mathfrak{m}^+$ onto a dense subset of X_c (that contains X_n) and the Remark follows because the requirement A) is fulfilled for the symmetric spaces. $\square$

4 The conjugate locus

Another way to reformulate the Proposition 1 is the following:

Theorem 1 *Let* $\widetilde{\mathbf{M}}$ *be a coherent state manifold with Hermitian symmetric space structure, parametrized in* $\mathcal{V}_0$ *around* $Z = 0$ *as in eqs. (2.14), (2.15). Then the conjugate locus of the point* o *is obtained vanishing the Jacobian of the exponential map* $Z = Z(B)$ *and the corresponding transformations of the chart from* $\mathcal{V}_0$.

Proof. The proof is contained in Propsition 2: *the dependence* $Z = Z(B)$, with $B \in \mathfrak{m}^+$, and Z parametrizing $\widetilde{\mathbf{M}}$, obtained passing from eq. (2.15) to (2.14) using (2.20) (the Baker-Campbell-Hausdorff formulas) [8], *expresses* in fact *the geodesic exponential* $\mathrm{Exp}_0 : \widetilde{\mathbf{M}}_0 \rightarrow \widetilde{\mathbf{M}}$. $\square$

The situation is very transparent in the case of the complex Grassmann manifold $X_c = G_n(\mathbb{C}^{n+m}) = SU(n+m)/S(U(n)\times U(m))$ and his noncompact dual $X_n = SU(n,m)/S(U(n)\times U(m))$. There [8]

$$
\begin{aligned}
X_{n,c} &= \mathrm{e}^{\begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix}} o = \begin{pmatrix} \mathrm{co}\sqrt{BB^*} & B\dfrac{\mathrm{si}\sqrt{B^*B}}{\sqrt{B^*B}} \\ \pm\dfrac{\mathrm{si}\sqrt{B^*B}}{\sqrt{B^*B}}B^* & \mathrm{co}\sqrt{B^*B} \end{pmatrix} o \\
&= \begin{pmatrix} \mathbb{1} & Z \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} (\mathbb{1}\mp ZZ^*)^{1/2} & 0 \\ 0 & (\mathbb{1}\mp Z^*Z)^{1/2} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \pm Z^* & \mathbb{1} \end{pmatrix} o \qquad (4.1)\\
&= \mathrm{e}^{\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}} P,
\end{aligned}
$$

where B^* denotes the hermitian conjugate of the matrix B. co is an abbreviation for the circular cosine cos (resp. the hyperbolic cosine coh) for X_c (resp. X_n) and similarly for si. The $-$ $(+)$ sign in the equation above corresponds to the compact (resp. noncompact) X.

Here Z and B are $n\times m$ matrices related by the relation

$$Z = B\frac{\mathrm{ta}\sqrt{B^*B}}{\sqrt{B^*B}}, \qquad (4.2)$$

and ta is an abbreviation for the hyperbolic tangent tgh (resp. the circular tangent tg) for X_n (resp. X_c). The dependence $Z = Z(B)$ describes in fact Exp : $G_n(\mathbb{C}^{n+m})_e \rightarrow G_n(\mathbb{C}^{n+m})$ in $\mathcal{V}_0$. Indeed, the equation of geodesics for $X_{c,n}$ is [13]

$$\frac{d^2Z}{dt^2} - 2\epsilon\frac{dZ}{dt}Z^+(\mathbb{1}+\epsilon ZZ^+)^{-1}\frac{dZ}{dt} = 0\ , \qquad (4.3)$$

where $\epsilon = 1$ (-1) for X_c (resp. X_n). It is easy to see that (4.2) verifies (4.3) with the initial condition $\dot{Z}(0) = B$.

Z and B in the eq. (4.2) of geodesics are in the same time the parameters describing the coherent states in the paramerization given by eq. (2.14) and respectively (2.15).

The following theorem summarizes the known facts about the tangent conjugate locus and conjugate locus in $G_n(\mathbb{C}^{m+n})$ [40, 41, 13]. The relevant fact for the present paper is that *the conjugate locus can be calculated using Theorem 1.* Below $\mathbf{O}^\perp$ denotes the orthogonal complement of the n-plane $\mathbf{O}$ in $\mathbb{C}^N$. We also use the notation

$$V_l^p = \left\{Z\in G_n(\mathbb{C}^{n+m}) | \dim(Z\cap\mathbb{C}^p)\geq l\right\}, \qquad (4.4)$$

$$W_l^p = V_l^p - V_{l+1}^p = \left\{Z\in G_n(\mathbb{C}^{n+m}) | \dim(Z\cap\mathbb{C}^p) = l\right\}, \qquad (4.5)$$

$$\omega_l^p = (\underbrace{p-l,\ldots,p-l}_{l},\underbrace{m,\ldots,m}_{n-l}). \qquad (4.6)$$

$$V_l^p = Z(\omega_l^p);\quad W_l^p = Z'(\omega_l^p)\ . \qquad (4.7)$$

The following sequences of integers are used

$$\omega = \{0 \le \omega(1) \le \ldots \le \omega(n) \le m\}, \tag{4.8}$$

$$\sigma(i) = \omega(i) + i, \;\; i = 1, \ldots, n. \tag{4.9}$$

The Schubert varieties are defined as

$$Z(\omega) = \left\{ X \in G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbb{C}^{\sigma(i)}) \ge i \right\}. \tag{4.10}$$

The "jumps" sequence is

$$I_\omega = \{0 = i_0 < i_1 < \ldots < i_{l-1} < i_l = n\} \;\;, \tag{4.11}$$

$$\omega(i_h) < \omega(i_{h+1}), \omega(i) = \omega(i_{h-1}), i_{h-1} < i \le i_h, h = 1, \ldots, l. \tag{4.12}$$

The generic elements of $Z(\omega)$ are

$$Z'(\omega) = \left\{ X \in G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbb{C}^{\sigma(i_h)}) = i_h, \;\; i_h \in I_\omega \right\}. \tag{4.13}$$

Theorem 2 *The tangent conjugate locus C_0 of the point* $\mathbf{O} \in G_n(\mathbb{C}^{m+n})$ *is given by*

$$C_0 = \bigcup_{k,p,q,i} ad\, k(t_i H) \;, \;\; i = 1, 2, 3; \;\; 1 \le p < q \le r, \;\; k \in K, \tag{4.14}$$

where the vector $H \in \mathfrak{a}$ is normalised,

$$H = \sum_{i=1}^{r} h_i D_{i\, n+i}, \;\; h_i \in \mathbb{R}, \;\; \sum h_i^2 = 1 \;. \tag{4.15}$$

The parameters t_i, $i = 1, 2, 3$ in eq. (4.14) are

$$\begin{aligned} t_1 &= \frac{\lambda \pi}{|h_p \pm h_q|} \;, \;\; \text{multiplicity } 2; \\ t_2 &= \frac{\lambda \pi}{2|h_p|} \;, \;\; \text{multiplicity } 1; \\ t_3 &= \frac{\lambda \pi}{|h_p|} \;, \;\; \text{multiplicity } 2|m-n|; \;\; \lambda \in \mathbb{Z}^\star \;. \end{aligned} \tag{4.16}$$

The conjugate locus of $\mathbf{O}$ *in* $G_n(\mathbb{C}^{m+n})$ *is given by the union*

$$\mathbf{C}_0 = \mathbf{C}_0^W \cup \mathbf{C}_0^I. \tag{4.17}$$

The following relations are true

$$\mathbf{C}_0^I = \exp \bigcup_{k,p,q} Ad\, k(t_1 H) \;, \tag{4.18}$$

$$\mathbf{C}_0^W = \exp\bigcup_{k,p} Ad\, k(t_2H)\ , \tag{4.19}$$

i.e. exponentiating the vectors of the type t_1H we get the points of $\mathbf{C}_0^I$ for which at least two of the stationary angles with $\mathbf{O}$ are equal, while the vectors of the type t_2H are sent to the points of $\mathbf{C}_0^W$ for which at least one of the stationary angles with $\mathbf{O}$ is 0 or $\pi/2$.

The $\mathbf{C}_0^W$ part of the conjugate locus is given by the disjoint union

$$\mathbf{C}_0^W = \begin{cases} V_1^m \cup V_1^n, & n \le m, \\ V_1^m \cup V_{n-m+1}^n, & n > m, \end{cases} \tag{4.20}$$

where

$$V_1^m = \begin{cases} \mathbb{CP}^{m-1}, & \text{for } n = 1, \\ W_1^m \cup W_2^m \cup \ldots W_{r-1}^m \cup W_r^m, & 1 < n, \end{cases} \tag{4.21}$$

$$W_r^m = \begin{cases} G_r(\mathbb{C}^{\max(m,n)}), & n \ne m, \\ \mathbf{O}^\perp, & n = m, \end{cases} \tag{4.22}$$

$$V_1^n = \begin{cases} W_1^n \cup \ldots \cup W_{r-1}^n \cup \mathbf{O}, & 1 < n \le m, \\ \mathbf{O}, & n = 1, \end{cases} \tag{4.23}$$

$$V_{n-m+1}^n = W_{n-m+1}^n \cup W_{n-m+2}^n \cup \ldots \cup W_{n-1}^n \cup \mathbf{O}\ ,\ n > m\ . \tag{4.24}$$

Sketch of the Proof (See [13]). The tangent conjugate locus C_0 for $G_n(\mathbb{C}^{m+n})$ in the case $n \le m$ was obtained by Sakai [41]. Sakai has observed that Wong's result on the conjugate locus in the manifold [40] is incomplete, i.e. $\mathbf{C}_0^W \subset \mathbf{C}_0$ but $\mathbf{C}_0^W \subsetneqq \mathbf{C}_0 = \exp C_0$. The proof of Sakai consists in solving the eigenvalue equation $R(X,Y^i)X = e_iY^i$ which appears when solving the Jacobi equation, where the curvature for the symmetric space $X_c = G_c/K$ at o is simply $R(X,Y)Z = [[X,Y],Z],\ X,Y,Z \in \mathfrak{m}_c$. Then $q = \mathrm{Exp}_0 tX$ is conjugate to o if $t = \pi\lambda/\sqrt{e_i},\ \lambda \in \mathbb{Z}^\star \equiv \mathbb{Z}\setminus\{0\}$.

Above $\mathfrak{a}$ is the Cartan subalgebra of the symmetric pair $(SU(n+m), S(U(n)\times U(m)))$ [38, 41, 13] consisting of vectors of the form (4.15) where r is the symmetric rank of X_c (and X_n) and we use the notation $D_{ij} = E_{ij} - E_{ji},\ i,j = 1,\ldots,N$. E_{ij} is the matrix with entry 1 on line i and column j and 0 otherwise. The results in the complex Grassmann manifold are obtained farther using the exponential map given by eq. (4.2).

The same result on the calculation of the tangent conjugate locus can be obtained [13] using Prop. 3.1 p. 294 in the book of Helgason [38]. This Proposition asserts that $H \in \mathfrak{a}$ is conjugate with o iff $\alpha(H) \in i\pi\mathbb{Z}^\star$ for some root α which do not vanishes identically on $\mathfrak{a}$. The eigenvalues of the equation $[H,X] = \lambda X,\ \forall H \in \mathfrak{a},\ X \in \mathfrak{g}^{\mathbb{C}}$ lead [13] to the values given in equation (4.14) for the parameters $t_1 - t_3$.

The direct proof [13] in the Grassmann manifold uses in Theorem 1 the dependence $Z = Z(B)$ furnished by eq. (4.2) which gives the geodesics on $G_n(\mathbb{C}^{n+m})$ and the Jordan's stationary angles between two $n-$planes.

The proof [13] is done in four steps. a) Firstly, a diagonalization of the $n\times m$ matrix Z is performed. b) Secondly, the Jacobian of a transformation of complex dimension one is computed. c) The cut locus is reobtained and his contribution to the conjugate locus is taken into account. d) The non-zero angles are counted using the following property of the stationary angles: if the n' (n)-plane $Z'_{n'}$ (resp. Z_n) are such that $Z'_{n'} \cap Z_n = Z"_{n"}$, than $n' - n"$ angles of $Z'_{n'}$ and Z_n are different from 0 and $n"$ are 0. $\square$

Comment 1 ${\mathbf{C_0}}^I$ *contains as subset the maximal set of mutually isoclinic subspaces of the Grassmann manifold, which are the isoclinic spheres, with dimension given by the solution of the Hurwitz problem.*

Proof. Wong [42] has found out the locus of isoclines in $G_n(\mathbb{R}^{2n})$, i.e. the maximal subset B of the Grassmann manifold containing $\mathbf{O}$ consisting of points with the property that every two n-planes of B have *all* the stationary angles equal. Two mutually isoclinic n-planes correspond to the situation where the matrix, which has as eigenvalues the squares of the stationary angles, is a multiple of $\mathbb{1}$. The results of Wong were generalized by Wolf [43], who has considered also the complex and quaternionic Grassmann manifolds. The problem of maximal mutually isoclinic subspaces is related with the Hurwitz problem [44]. Any maximal set of mutually isoclinic n-planes is analytically homeomorphic to a sphere (cf. Thm 8.1 in Wong [42] and Wolf [43]), the dimension of the isoclinic spheres being given by the solution to the Hurwitz problem. $\square$

5 The cut locus

Let X be complete Riemannian manifold. The point q is in the *cut locus* $\mathbf{CL_p}$ of $p \in X$ if q is the nearest point to p on the geodesic emanating from p beyond which the geodesic ceases to minimize arc length (cf. [33], see also Ref. [11] for more references).

Remark 1 $\operatorname{codim}_{\mathbb{C}} \mathbf{CL}_p \geq 1$.

We call *polar divisor* of $\boldsymbol{e}_0$ the set $\Sigma_0 = \{\boldsymbol{e} \in e(G) | (\boldsymbol{e}_0, \boldsymbol{e}) = 0\}$. This denomination is inspired after Wu [45], who used this term in the case of the complex Grassmann manifold $G_n(\mathbb{C}^{m+n})$.

Theorem 3 *Let* $\widetilde{\mathbf{M}}$ *be a homogeneous manifold* $\widetilde{\mathbf{M}} \approx G/K$. *Suppose that there exists a unitary irreducible representation* π_j *of* G *such that in a neighbourhood* $\mathcal{V}_0$ *around* $Z = 0$ *the coherent states are parametrized as in eq. (2.14). Then the manifold* $\widetilde{\mathbf{M}}$ *can be represented as the disjoint union*

$$\widetilde{\mathbf{M}} = \mathcal{V}_0 \cup \Sigma_0. \tag{5.1}$$

Moreover, if the condition B) is true, then

$$\Sigma_0 = \mathbf{CL}_0. \tag{5.2}$$

Proof. We can take $\psi = \psi(Z) = \boldsymbol{e}_Z \in \mathbf{M}$ such that the parameters Z are in $\mathbb{C}^n$ as in formula (2.14). Now, the second relation (2.20) implies that $(0, \psi) = 1$ for $\psi \in \xi_0^{-1}(\mathcal{V}_0)$. It follows that the equation

$$\cos\theta = 0, \tag{5.3}$$

where

$$\cos\theta = \frac{|(0,\psi)|}{\|0\|^{1/2}\|\psi\|^{1/2}} = \|\psi\|^{-1/2}, \tag{5.4}$$

does not have solutions for $\psi \in \xi_0^{-1}(\mathcal{V}_0)$, and the representation (5.1) follows.

To prove relation (5.2) if B) is true, use is made of Thm. 7.4 and the subsequent remark at p. 100 from Ref. [33]. The theorem essentially says that any Riemannian manifold $\widetilde{\mathbf{M}}$ is the disjoint union of the cut locus (closed cell) and the largest open cell of $\widetilde{\mathbf{M}}$ on which normal coordinates can be defined. But $Z \in \mathbb{C}^n$ for points of $\mathcal{V}_0$ corresponding to the largest normal coordinates $B \in \mathfrak{m}$, because B) implies A). □

Corollary 1 *Suppose that $\widetilde{\mathbf{M}}$ verifies B) and admits the embedding (2.6). Let $0, Z \in \widetilde{\mathbf{M}}$. Then $Z \in \mathbf{CL}_0$ iff the Cayley distance between the images $\iota(0), \iota(Z) \in \mathbf{P}(\mathcal{H})$ is $\pi/2$*

$$d_c(\iota(0), \iota(Z)) = \pi/2. \tag{5.5}$$

Here d_c denotes the the hermitian elliptic Cayley distance on the projective space

$$d_c([\boldsymbol{\omega}'],[\boldsymbol{\omega}]) = \arccos\frac{|(\boldsymbol{\omega}',\boldsymbol{\omega})|}{\|\boldsymbol{\omega}'\|\|\boldsymbol{\omega}\|}\ . \tag{5.6}$$

Comment 2 *The cut locus is present everywhere one speaks about coherent states on symmetric spaces.*

This assertion is largely explicated in reference [14], where there are discussed the consequences of theorem 3 for the references [30, 5, 31, 46, 6].

We remember the explicit expression of the cut locus on the complex projective space and Grassmannian.

Remark 2 *On $\mathbb{CP}^n$, $\mathbf{CL}_0 = \Sigma_0 = H_1 = \mathbb{CP}^{n-1}$.*

Proof. Let the notation $\mathcal{V}_i = \{z \in \mathcal{H}^* | z_i \neq 0\}$, $\mathcal{U}_i = \xi(\mathcal{V}_i)$, $H_i = \mathbf{P}(\mathcal{H}) \setminus \mathcal{U}_i$. The point $p_0 = [1, 0, 0, \ldots] \in \mathbf{P}(\mathcal{H})$ corresponds to the point 0 in the Remark. Then the solution of the equation $([p_0],[z]) = 0$ is $[z] = [0, \times, \times, \ldots] = H_1 = \mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ for $\mathcal{H} = \mathbb{C}^n$.□

Proposition 3 (Wong [47]) *The cut locus of the point $\mathbf{O}$ is given by*

$$\begin{aligned} \mathbf{CL}_0 &= \Sigma_0 = V_1^m = Z(\omega_1^m) = Z(m-1, m, \ldots, m) \\ &= \left\{X \subset G_n(\mathbb{C}^{n+m}) | \dim(X \cap \mathbf{O}^{\perp}) \geq 1\right\}. \end{aligned} \tag{5.7}$$

The cut locus in $G_n(\mathbb{C}^{m+n})$ is given by those $n-$planes which have at least one of the stationary angles $\pi/2$ with the $n-$plane $\mathbf{O}$.

Proof. An immediate proof can be obtained using the results of Wu referring to the polar divisor Σ_0 on the Grassmann manifold (see Ch. 1 in Ref. [45]) and the theorems characterising the canonical (universal, det) bundle on $G_n(\mathbb{C}^N)$ (see especially Prop. 3.3 Ch. 7 in Ref. [21]), which are particularisations of the representation in Thm. 3. □

6 The divisors

Proposition 4 *If $\widetilde{\mathbf{M}}$ is an homogenous algebraic manifold, then the polar divisor $\Sigma_0 = \iota^* H_1$ is a divisor.*

Proof. This fact is a consequence of standard properties of divisors [48] (cf. [16]). □

The result is true for any algebraic Kähler manifold [18].

Let [ ] be the functorial homomorphism between the group of divisors and the Picard group of equivalence class of holomorphic line bundles [48]

$$[\ ] : \mathrm{Div}(\widetilde{\mathbf{M}}) \stackrel{\delta^0}{\rightarrow} H^1(\widetilde{\mathbf{M}}, \mathcal{O}^*). \tag{6.1}$$

Theorem 4 *Let $\widetilde{\mathbf{M}}$ be a simply connected Hodge manifold admitting the embedding (2.6). Let $\mathbf{M} = \iota^*[1]$ be the unique, up to equivalence, projectively induced line bundle with a given admissible connection. Then $\mathbf{M} = [\Sigma_0]$. Moreover, if the homogeneous manifold $\widetilde{\mathbf{M}}$ verifies condition B), then $\mathbf{M} = [\mathbf{CL}_0]$. In particular, the first relation is true for Kählerian C-spaces, while the second one for hermitian symmetric spaces.*

Proof. The main part of the proof is based on the following theorem of Kodaira and Spencer: For an algebraic manifold there is a isomorphism of the group $\mathrm{Cl}(\widetilde{\mathbf{M}})$ of divisor classes with respect to linear equivalence with the Picard group $\mathrm{Pic}(\widetilde{\mathbf{M}})$, i.e. for every complex line bundle $\mathbf{M}$ over an algebraic manifold $\widetilde{\mathbf{M}}$ there exists a divisor D such that $[D] = \mathbf{M}$ (cf. [49]). The next ingredient is the following theorem due to Kostant: Let $\widetilde{\mathbf{M}}$ be a simply connected Hodge manifold. Then, up to equivalence, there exists a unique line bundle with a given curvature matrix $\Theta_{\mathbf{M}}$ of the hermitian connection, or, equivalently, with a given admissible connection (Thm. 2.2.1 in [50] p. 135). Farther the theorem 3 is used. The information on Kählerian $C-$ spaces is extracted from [28, 17]. □

In the same context, we remember *Bertini's theorem:* Let $\mathbf{M}$ be a projectively induced line bundle over an algebraic manifold $\widetilde{\mathbf{M}}$. Then there is a *non-singular* divisor D of $\widetilde{\mathbf{M}}$ with $\mathbf{M} = [D]$. Another formulation of Bertini's theorem is: A general hyperplane section S of a connected non-singular algebraic manifold $\widetilde{\mathbf{M}}$ in $\mathbb{CP}^N$ is itself non-singular and for $n \geq 2$, connected (cf. Zariski [51], Akizuki [52], Hartshorne [53]).

Comment 3 *Generally, the divisor Σ_0 is singular because it doesn't corresponds to a general section in Bertini's theorem.*

Proof. We illustrate the assertion on the example furnished by the Grassmannian $G_2(\mathbb{C}^4)$. The Plücker embedding is $G_2(\mathbb{C}^4) \hookrightarrow \mathbb{CP}^5$. There are six coordinate neighbourhoods $\mathcal{V}_1 - \mathcal{V}_6$. In $\mathcal{V}_6$: $p_{12} = 0; p_{14}p_{23} - p_{13}p_{24} = 0$. This is a cone over a quadric surface whose vertex is the point $(0,0,0,0,0,1)$. The hyperplane $p_{12} = 0$ is the embedded tangent hyperplane of $G_2(\mathbb{C})^4$ of the line $x_1 = x_2 = 0$ in $\mathbb{CP}^5$. A general hyperplane section of $G_2(\mathbb{C})^4$ is not of the form $p_{12} = 0$, since by Bertini's theorem it has to be smooth. See details in [16, 54].□

7 The Calabi's diastasis

We remember that the *Calabi's diastasis* [55] is expressed through the coherent states (cf. [6]) as

(7.1) $$D(Z', Z) = -2\log|(\underline{e}_{Z'}, \underline{e}_Z)|.$$

In proving Proposition 6 the following Proposition [9, 15, 12] is needed.

Proposition 5 *Let $\underline{\boldsymbol{e}}_Z$ as in (2.14), where Z parametrizes the coherent state manifold in the $\mathcal{V}_0 \subset \widetilde{\mathbf{M}}$ and let us suppose that the coherent state manifold admits the kählerian embedding (2.6). Then the angle*

(7.2) $$\theta := \arccos|(\underline{\boldsymbol{e}}_{Z'}, \underline{\boldsymbol{e}}_Z)|,$$

is equal to the Cayley distance on the geodesic joining $\iota(Z'), \iota(Z)$, where $Z', Z \in \mathcal{V}_0$,

(7.3) $$\theta = d_c(\iota(Z'), \iota(Z)).$$

More generally, it is true the following relation (Cauchy formula)

(7.4) $$(\underline{\boldsymbol{e}}_{Z'}, \underline{\boldsymbol{e}}_Z) = (\iota(Z'), \iota(Z)).$$

Proof. We discuss firstly the case of compact manifolds. Let $i : \widetilde{\mathbf{M}} \to \mathfrak{H}$ the mapping $i(Z) = \boldsymbol{e}_Z$. Then the embedding (2.6) is realised effectively by the formula $\iota = \xi \circ i$, i.e.

(7.5) $$\iota(Z) = [\underline{\boldsymbol{e}}_Z]\ .$$

Since the manifold $\widetilde{\mathbf{M}}$ admits the embedding (2.6), the line bundle $\mathbf{M}$ is a positive one. The following theorem [56, 48, 19, 57] is applied: Let $\mathbf{M}'$ be a holomorphic line bundle on a compact complex manifold $\widetilde{\mathbf{M}}$. The following conditions are equivalent: a) $\mathbf{M}'$ is positive; b) for all coherent analytic sheaves $\mathcal{S}$ on $\widetilde{\mathbf{M}}$ there exists a positive integer $m_0(\mathcal{S})$ such that $H^i(\widetilde{\mathbf{M}}, \mathcal{S} \otimes \mathbf{M}'^m) = 0$ for $i > 0$, $m \geq m_0(\mathcal{S})$ (the vanishing theorem of Kodaira); c) there exists a positive integer m_0 such that for all $m \geq m_0$, there is an embedding $\iota_{\mathbf{M}} : \widetilde{\mathbf{M}} \hookrightarrow \mathbb{CP}^{N-1}$ for some $N \geq D$ such that $\mathbf{M} = \mathbf{M}'^m$ is projectively induced, i.e. $\mathbf{M} = \iota^*[1]$; d) $\widetilde{\mathbf{M}}$ is a Hodge manifold (the embedding theorem of Kodaira); e) the fundamental two-form of $\widetilde{\mathbf{M}}$, the curvature matrix and the first Chern class of $\mathbf{M}'$ are related by the relations $\omega = \frac{\sqrt{-1}}{2}\Theta_{\mathbf{M}'}$, $c_1(\mathbf{M}') = \frac{\omega}{\pi}$. f) moreover, if $\widetilde{\mathbf{M}}$ is a kählerian C-space, then $\widetilde{\mathbf{M}}$ is a flag manifold.

But the line bundle $\mathbf{M}$ is already very ample and the holomorphic map $\iota_{\mathbf{M}} : \widetilde{\mathbf{M}} \hookrightarrow \mathbb{CP}^{N-1}$ given by

(7.6) $$\iota_{\mathbf{M}} = [s_1(m), \ldots, s_N(m)]$$

is a holomorphic embedding, where $s_1(m), \ldots, s_N(m) \in \Gamma(\widetilde{\mathbf{M}}, \mathbf{M})$ are global sections. Remark that eq. (7.5) realizes the embedding (7.6).

The very ample holomorphic line bundle $\mathbf{M}$ of coherent vectors is the pull-back ι^* of the hyperplane bundle [1] of $\mathbf{P}(\mathfrak{H})$, the dual bundle of the tautological line bundle on $\mathbf{P}(\mathfrak{H})^*$, i.e. $\mathbf{M} = \iota^*[1]$. The analytic line bundle $\mathbf{M}$ is *projectively induced* (see p. 139 in Ref. [58]). Then the mapping i preserves the scalar product

$$(\boldsymbol{e}_{Z'}, \boldsymbol{e}_Z) = (i(Z'), i(Z)), \tag{7.7}$$

which imply

$$(\underline{\boldsymbol{e}}_{Z'}, \underline{\boldsymbol{e}}_Z) = \frac{(\iota(Z'), \iota(Z))}{\|\iota(Z')\|\|\iota(Z)\|}, \tag{7.8}$$

and formula (7.4) follows. Eq. (7.3) follows from eq. (7.8) and the definition (5.6) of the Cayley distance.

The noncompact case is treated similarly [59]. □

Proposition 6 *The diastasis distance $D(Z', Z)$ between $Z', Z \in \mathcal{V}_0 \subset \widetilde{\mathbf{M}}$ is related to the geodesic distance $\theta = d_c(\iota(Z'), \iota(Z))$, where ι is the embedding (2.6), by the relation*

$$D(Z', Z) = -2 \log \cos \theta.$$

If $\widetilde{\mathbf{M}}_n$ is noncompact, $\iota' : \widetilde{\mathbf{M}}_n \hookrightarrow \mathbb{CP}^{N-1,1} = SU(N,1)/S(U(N) \times U(1))$, and $\delta_n(\theta_n)$ is the length of the geodesic arc joining $\iota'(Z'), \iota'(Z)$ (resp. $\iota(Z'), \iota(Z)$), then

$$\cos \theta_n = (\cosh \delta_n)^{-1} = e^{-D/2}.$$

Proof. The Proposition is a direct consequence of the relation (7.1) and of the Proposition 5.

Comment 4 *The relation (5.2) furnishes for manifolds of symmetric type, i.e. verifying condition B), a geometric description of the domain of definition of Calabi's diastasis: for z fixed, $z' \notin \mathbf{CL}_z$.*

8 The Euler-Poincaré characteristic, the Borel-Morse cells, Kodaira embedding,...

Theorem 5 *For flag manifolds $\widetilde{\mathbf{M}} \approx G/K$, the following 7 numbers are equal:*

1) the maximal number of orthogonal coherent vectors;

2) the number of holomorphic global sections of the holomorphic line bundle $\mathbf{M}$ over $\widetilde{\mathbf{M}}$, which is supposed to be very ample;

3) the dimension of the fundamental representation in the Borel-Weil theorem;

4) the minimal N appearing in the Kodaira embedding theorem, $\iota : \widetilde{\mathbf{M}} \hookrightarrow \mathbb{CP}^{N-1}$;

5) the number of critical points of the energy function f_H attached to a Hamiltonian H linear in the generators of the Cartan algebra of G, with unequal coefficients;

6) the Euler-Poincaré characteristic $\chi(\widetilde{\mathbf{M}}) = [W_G]/[W_H]$, $[W_G] = card\, W_G$, where W_G denotes the Weyl group of G;

7) the number of Borel-Morse cells which appear in the CW-complex decomposition of $\widetilde{\mathbf{M}}$.

Proof. The equivalence 1) with 2) follows from the definition of coherent states as holomorphic global sections in the $G-$homogenous line bundle M. The Weyl group interchanges the origines in the different coordinate neighbourhoods of the manifold $\widetilde{\mathbf{M}}$.

Then 2) is equivalent with 3) due to Borel-Weil theorem which essentially asserts that [29]: For every irreducible representation π_j of dominant weight j of the compact connected semisimple Lie group G corresponds on every homogenous Kählerian space $G/K \approx G^{\mathbb{C}}/P_j$ a complete linear system $|D|$. The representation space $\mathfrak{H}_j$ of the representation π_j is the dual of $\mathfrak{L}(D)$. The associated line bundle $\mathbf{M}'$ is ample iff the space $G/K \approx G^{\mathbb{C}}/P_j$ is strictly associated to the representation $\tilde{\pi}_j$. Note that $\mathbf{M}' = \mathbf{M}$, because we have supposed that the line bundle $\mathbf{M}$ is already very ample.

The equivalence of 1) with 4) follows from the realization of the embedding (2.6) under the form (7.6) given by eq. (7.5).

In order to prove the equivalence of 1) with 5), the energy function (covariant Berezin symbol [30]) $f_H : \widetilde{\mathbf{M}} \rightarrow \mathbb{R}$

$$f_H(Z, \overline{Z}) = (\underline{\boldsymbol{e}}_Z, H\underline{\boldsymbol{e}}_Z) \tag{8.1}$$

is attached to the Hamiltonian

$$H = \sum_{i=1}^{r} \epsilon_i H_i,\ \epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \mathbb{R}. \tag{8.2}$$

The operators H_i are defined in eq. (2.20). Then it was proved in Reference [7] that the energy function (8.1) attached to the Hamiltonian (8.2) is a Perfect Morse function in the extended sense.

The equivalence of 1) with 6) is contained in Theorem 2) in the same reference [7]. Here we stress only that $\chi(G/K) > 0$ if and only if Rank $G =$ Rank K [60], i.e. $\widetilde{\mathbf{M}}$ is a flag manifold. The same Theorem 2) in [7] implies the equivalence of 1) and 7).□

Comment 5 *The Weil prequantization condition is nothing else that the condition to have a Kodaira embedding, i.e. the algebraic manifold* $\widetilde{\mathbf{M}}$ *to be a Hodge one.*

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Institute for Physics and Nuclear Engineering
Department of Theoretical Physics
PO BOX MG-6, Bucharest-Magurele, Romania
E-mail: Berceanu@theor1.ifa.ro; Berceanu@Roifa.Ifa.Ro