A Generalization for Peakon's Solitary Wave and Camassa-Holm Equation

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Abstract

A class of new integrable evolution equations, generalizing the Camassa-Holm equation, is obtained. They admit special solutions of peakon's type (with a corner at the peak of one of the higher order derivatives). The new equations will have a bi-Hamiltonian structure each one, so they are all completely integrable in the sense of Arnold-Liouville, and a common isospectral representation.

1 Introduction

It has been recently pointed out that the equation:

(1) $$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

derived by Camassa and Holm (see [5] and [6]), is completely integrable in the sense of Arnold-Liouville: if we write $m = u - u_{xx}$, then the equation (1) takes the form:

(2) $$m_t = -\mathcal{E}\frac{\delta\mathcal{H}_2}{\delta m} = -\mathcal{D}\frac{\delta\mathcal{H}_1}{\delta m},$$

where

(3) $$\left\{ \begin{array}{lcl} \mathcal{D} & = & (m+\kappa)\partial_x + \partial_x(m+\kappa) \\ \mathcal{E} & = & \partial_x + {\partial_x}^3 \end{array} \right.$$

are two compatible Hamiltonian operators, and

(4) $$\left\{ \begin{array}{lcl} \mathcal{H}_1 & = & \frac{1}{2}\int_{-\infty}^{+\infty}(u^2 + {u_x}^2)dx \\ \mathcal{H}_2 & = & \frac{1}{2}\int_{-\infty}^{+\infty}(u^3 + uu_{xx}^2 - \kappa u^2)dx. \end{array} \right.$$

Furthermore, for $\kappa = 0$, the equation (1) has a special solitary wave solution with a corner at its peak, named "peakon"; this takes the simple form:

(5) $$u(x,t) = c\, e^{-|x-ct|}.$$

More generally, Camassa and Holm showed that

(6) $$u(x,t) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}$$

[1] Although his participation to both the Workshop and the Seminar was previously announced, the author couldn't attend these scientific events due to reasons beyond his control

gives an N-peakon solution, provided q_i and p_i solves the Hamilton canonical system:

$$(7) \qquad \begin{cases} \dot{q}_j &= \sum_{k=1}^{N} p_k \, e^{-|q_j - q_k|} \\ \dot{p}_j &= -p_j \sum_{k=1}^{N} p_k \, sgn(q_j - q_k), e^{-|q_j - q_k|} \end{cases}$$

with Hamiltonian H_A:

$$H_A = \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|}.$$

Later, F. Calogero and J.-P. Françoise proved in [3] and [4] that the more generally dynamical system characterized by the Hamiltonian:

$$H = \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j f(q_i - q_j).$$

is completely integrable in the Liouville sense for

$$f(x) = \lambda + \mu \cos(\nu x) + \mu' \sin(\nu |x|) ,$$

with λ, μ, μ' and ν being arbitrary constants. Here λ and μ are real, μ' and ν both real or both imaginary. For $\lambda = 0$, $\mu = 1$, $\mu' = i$ and $\nu = i$, the function f is precisely

$$f(x) = e^{-|x|} ,$$

so the Hamiltonian H will be the Camassa -Holm's Hamiltonian H_A.

The purpose of this paper is to point out a set of integrable evolution equations, generalizing the Camassa-Holm equation, such that each one admits special solutions of peakon's type (one of the higher order derivatives is discontinuous). By construction, the new equations will have a bi-Hamiltonian structure each one, and the same isospectral Lax representation as for the Camassa -Holm equation.

2 Construction of the generalized peakon

We consider an ordinary differential equation

$$(8) \qquad a_0 f + a_1 f'' + ... + a_n f^{(2n)} = 0,$$

where $n \geq 2$ and $(a_0, a_1, ..., a_n)$ is a set of (real) constants such that the equation (8) has at least a solution f_0, verifying the conditions

$$f_0'(0) = f_0''(0) = \dots = f_0^{(2n-3)} = 0, \tag{9}$$

$$f_0^{(2n-1)}(0) = \beta \neq 0 \tag{10}$$

and

$$f_0 \to 0\,,\; x \to \infty. \tag{11}$$

(The existence of at least one of this kind of solution, for each n, is assured by the classical theory of differential equations with constant coefficients.)

We put

$$g(x) = \begin{cases} f_0(x) & , \quad x \geq 0 \\ f_0(-x) & , \quad x < 0 \end{cases} \tag{12}$$

and we see that g verifies the equation (8) on $\mathbb{R} - \{0\}$, $g \in \mathcal{C}^{2n-2}(\mathbb{R})$ and

$$\begin{cases} g'(0) = g''(0) = \dots = g^{(2n-3)} = 0, \\ g^{(2n-1)}(0+0) = -f^{(2n-1)}(0-0) = \beta \neq 0, \\ g \to 0 \, when \, |x| \to \infty. \end{cases} \tag{13}$$

The distribution's derivation of singular functions formulas gives:

$$a_0 g + a_1 g'' + \dots + a_n g^{(2n)} = 2a_n \beta \delta(x), \tag{14}$$

because g has a singularity of the $(2n-1)$'th derivative in 0.

We look for an evolution equation verified by the function

$$u(x,t) = \sum_{i=1}^{N} p_i(t) g(x - q_i(t)) \tag{15}$$

where $\{p_i\ ,\ q_i\}$ solves the Hamiltonian system of Calogero type:

$$\begin{cases} \dot{q}_j & = & \displaystyle\sum_{k=1}^{N} p_k \; g(q_j - q_k) \\ \dot{p}_j & = & -p_j \displaystyle\sum_{k=1}^{N} p_k \, g'(q_j - q_k) \end{cases} , \tag{16}$$

for any N. (A Hamiltonian function for this system (16) will be

$$H_c = \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j \; g(q_i - q_j)\,.\,)$$

Let's have the notation

$$m(x,t) = a_0 g + a_1 g'' + \dots + a_n g^{(2n)}. \tag{17}$$

If u has the expresion (15), this means

$$m(x,t) = 2a_n\beta\sum_{i=1}^{N} p_i\ \delta(x). \tag{18}$$

Some computation gives:

$$\begin{aligned} m_t &= 2a_n\beta\,[\sum_{i=1}^{N}\dot{p}_i(t)\ \delta(x-q_i(t)) - \sum_{i=1}^{N} p_i(t)\ \delta'(x-q_i(t))\dot{q}_i(t)] \\ &= -2a_n\beta\sum_{i,j=1}^{N} p_ip_j[g'(q_i-q_j)\ \delta(x-q_i) + \\ &+\ g(q_i-q_j)\ \delta'(x-q_i)] \end{aligned} \tag{19}$$

(δ' is the derivative in the sense of distributions of the Dirac function, and all the equations in m will be considered on the space of zero degree distributions. This space can be canonically identified with the Radon-measures space - the zero degree distributions over ${\mathcal{C}_0}^0(\mathbb{R})$ - , so the multiplications with a $\mathcal{C}^1$-function are correctly defined.)

We note that

$$\begin{cases} m_xu &= 2a_n\beta\displaystyle\sum_{i,j=1}^{N} p_ip_jg(x-q_j)\ \delta'(x-q_i), \\ mu_x &= 2a_n\beta\displaystyle\sum_{i,j=1}^{N} p_ip_jg'(x-q_j)\ \delta(x-q_i). \end{cases} \tag{20}$$

Now, if ϕ is a test function, we have:

$$\begin{aligned} \langle g(x-q_j)\ \delta'(x-q_i)\ ,\phi(x)\rangle &= -\langle\delta(x-q_i)\ ,[g(x-q_j)\ \phi(x)]'\rangle \\ &= -g(q_i-q_j)\ \phi'(q_i) - g'(q_i-q_j)\ \phi(q_i) \\ \langle g'(x-q_j)\ \delta(x-q_i)\ ,\phi(x)\rangle &= \langle\delta(x-q_i)\ ,g'(x-q_j)\phi(x)\rangle \\ &= g'(q_i-q_j)\ \phi(q_i). \end{aligned}$$

Therefore

$$\langle m_t,\phi\rangle = -\langle m_xu,\phi\rangle - 2\langle mu_x,\phi\rangle \tag{21}$$

for any test function ϕ, that is the function $m(x,t)$ verifies the equation:

$$\mathbf{m_t = -m_xu - 2mu_x}. \tag{22}$$

Expressing only in function of the dependent variable u, the desired equation will be

$$\sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{a_k}\left(\mathbf{u^{(2k)}}\right)_{\mathbf{t}} + \mathbf{3a_0uu_x} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{a_k}\left(\mathbf{uu^{(2k+1)}} + \mathbf{u_x u^{(2k)}}\right) = \mathbf{0}, \tag{23}$$

where, conventionally

$$u^{(k)} = \frac{\partial^k}{\partial x^k} u.$$

In conclusion, for any $n \geq 2$ we have a generalization of the Camassa-Holm equation, admitting the special solitary wave solution:

$$u(x,t) = c\, g(x - ct) \tag{24}$$

(in the case $N = 1$), generalization of the Camassa-Holm peakon. The corresponding N-peakon solution was:

$$u(x,t) = \sum_{i=1}^{N} p_i(t) g(x - q_i(t)), \tag{25}$$

so our solitary wave (24) is a soliton in the classical sense.

3 The integrability of the new equations

We consider here the equation (23) on the real line, either with u and its derivatives vanishing at infinity.

Motivated by the bi-Hamiltonian structure of the equation (1), we put

$$\mathcal{H}_1 := \frac{1}{2} \int_{-\infty}^{+\infty} \sum_{k=0}^{n} (-1)^k a_k \left(u^{(k)}\right)^2 dx \tag{26}$$

and

$$m := \frac{\delta \mathcal{H}_1}{\delta u} = a_0 u + a_1 u'' + ... + a_n u^{(2n)} \tag{27}$$

(m is the momentum canonically conjugate to u, so $\frac{\delta \mathcal{H}_1}{\delta m} = u$).

Therefore, (22) has the Lie-Poisson expression:

$$m_t = -(m\partial + \partial m) \frac{\delta \mathcal{H}}{\delta m}. \tag{28}$$

Remark. The (28) form is the same as the first Poisson structure for the Camassa-Holm equation (1) and it is a generalization of the momentum part of Green-Naghdi or Boussinesq equations [6].

Note that the differential operator

$$\mathcal{D} = m\partial_x + \partial_x m \tag{29}$$

is a Hamiltonian operator (see [6], or [8], [10]- last chapter). Therefore (28) is effectively a Lie-Poisson structure.

For the second Hamiltonian structure, let us take:

$$\mathcal{H}_2 := \frac{1}{2}\int_{-\infty}^{+\infty} a_0 u^3 + \sum_{k=1}^{n} a_k (-1)^k u \big(u^{(k)}\big)^2 dx. \tag{30}$$

Then

$$m_t = -\partial \frac{\delta \mathcal{H}_2}{\delta u} = -\left(\sum_{k=0}^{n} a_k \partial^{2k+1}\right) \frac{\delta \mathcal{H}_2}{\delta m}$$

and consequently :

$$m_t = -\mathcal{E}\frac{\delta \mathcal{H}_2}{\delta m}, \tag{31}$$

where $\mathcal{E} = \sum_{k=0}^{n} a_k \partial^{2k+1}$ is a differential operator with constant coefficients, hence a Hamiltonian operator (see [10]).

The two Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ founded before, so that

$$m_t = -\mathcal{E}\frac{\delta \mathcal{H}_2}{\delta m} = -\mathcal{D}\frac{\delta \mathcal{H}_1}{\delta m}, \tag{32}$$

are compatible (this means that their sum or any other linear combination is still a Hamiltonian operator - see [9] for definitions and [10] for a simple characterization of this property). They define then a *recursion* operator $\mathcal{R} = \mathcal{D}\mathcal{E}^{-1}$ which leads to a collection of commuting flows:

$$\begin{aligned}
m_{(s+1),t} &:= \mathcal{K}_{(s+1)} \\
\mathcal{K}_{(s+1)} &:= -\mathcal{E}\frac{\delta \mathcal{H}_s}{\delta m} \\
&:= -\mathcal{D}\frac{\delta \mathcal{H}_{s-1}}{\delta m} \\
&= \mathcal{R}\mathcal{K}_{(s)} && (33) \\
\mathcal{K}_{(s+1)} &= \mathcal{R}\mathcal{K}_{(s)} \text{ (the recursion relation)}, && (34)
\end{aligned}$$

where $\mathcal{H}_n$ $(s \geq 3)$ is recursively obtained by the Magri's theorem (in [9]).

The recursion relation can be applied also for $s \leq 1$. The relation:

$$\{\mathcal{H}_r, \mathcal{H}_s\} = 0 \tag{35}$$

is verified for any integers r, s , so we have an infinite collection of conserved quantities [10] in involution for the initial equation (23) (which leads for $s = 2$). By definition we say that (23) is *completely integrable in the sense of Arnold-Liouville.*

For $s = 1$ we have

$$\begin{aligned} m_t &:= -(\sum_{k=0}^{n} a_k \partial^{2k+1})\frac{\delta \mathcal{H}_1}{\delta m} \\ &= -m_x \\ &:= -(m\partial + \partial m)\frac{\delta \mathcal{H}_0}{\delta m} \end{aligned}$$

so that

$$\mathcal{H}_0 := \frac{1}{2}\int_{-\infty}^{+\infty} m dx \tag{36}$$

(the total momentum).

For $s = 0$, $m_t = 0$. We can put

$$\mathcal{H}_{-1} := \int_{-\infty}^{+\infty} \sqrt{m} dx, \tag{37}$$

because $\mathcal{H}_{-1}$ is a Casimir function for the Hamiltonian operator $\mathcal{D}$. Then the recursion relation can be continued for s negative.

Remark. A supplementary condition like "m has no zeros and is positive" must be supposed in order that we can hope to find conservation laws $\mathcal{H}_s$ with $s < 0$.

Note that for $s = -1$ the flow

$$m_{(0),t} = -\mathcal{E}\frac{\delta \mathcal{H}_1}{\delta m_{(0)}} = -(\sum_{k=0}^{n} a_k \partial^{2k+1})\frac{1}{2\sqrt{m_{(0)}}} \tag{38}$$

is large generalization of the flow of completely integrable Harry-Dym equation [1]:

$$m_t = \partial^3 \frac{1}{2\sqrt{m}}. \tag{39}$$

The Hamiltonian operator pair for the Harry-Dym equation is [9]:

$$\begin{aligned} \mathcal{E}_D &= -\partial^3, \\ \mathcal{D}_D &= -(m\partial + \partial m), \end{aligned}$$

so the hierarchy defined by (23) is a generalization of the hierarchy of Harry-Dym equation.

4 The associated isospectral problem

For certain nonlinear equations, the Lax method associates to the considered equation a linear operator of Sturm-Liouville type so that the eigenvalues of the later are integrals of the former. Let us point out how one comes up with the isospectral problem associated to (23).

We consider the Schrödinger operator

$$L = -{\partial_x}^2 + V(x,t,\lambda), \tag{40}$$

where λ is a parameter. (In the classical case of the KdV equation, λ appears as a eigenvalue: $V = u - \lambda$.)

We look for a solution A of *the Lax system*:

$$\begin{cases} L\psi & = & 0, \\ (L_t + [L,A])\psi & = & 0. \end{cases} \tag{41}$$

Here A is a linear operator (in x), which may depend on $u(x,t)$, L_t is the derivative with respect to the parameter t as it appears explicitly in the operator L and $[L,A] = LA - AL$ is the commutator acting on some appropiate functions space.

We wish to express the equation (23) in the *Lax form*:

$$L_t + [L,A] = 0. \tag{42}$$

L being self-adjoint, A is to be anti-symmetric and there is a natural choice to construct it from a linear combination of odd derivatives in x.

Further, A is required to have enough freedom in any unknown constants or functions to enable the operator $L_t + [L,A]$ to be chosen of degree zero (an operator of multiplication).

The simplest choice is:

$$A = a(x,t)\partial_x + b(x,t). \tag{43}$$

We find:

$$V(x,t,\lambda) = \frac{1}{2\lambda}m + \frac{1}{4}, \tag{44}$$

$$\begin{cases} a & = & \lambda - u \\ b & = & -\frac{1}{2}u_x + \gamma. \end{cases} \tag{45}$$

(γ is an arbitrary constant).

So, equivalently, the equation (22) can be written like the compatibility condition (see [1]):

$$\psi_{xxt} = \psi_{txx} \tag{46}$$

between a second order *non-homogenous* spectral problem of Sturm-Liouville type:

$$\psi_{xx} - \frac{1}{4}\psi = -\frac{1}{2\lambda}m\psi \tag{47}$$

and a *linear* evolution equation:

$$\psi_t = (\lambda - u)\psi_x + (\frac{u_x}{2} + \gamma)\psi. \tag{48}$$

Note that if (λ, ψ) verifies the equation (47) for an initial condition $m(x, 0)$, then λ will be an integral for our equation (22) :

$$\frac{\partial}{\partial t}\lambda(m(x,t)) = 0. \tag{49}$$

Remark. The isospectral representations (47) and (48)are the same as those for the Camassa-Holm equation (1) and are independent of the choice of the equation (8).

Final remark. For each $n \geq 2$, we proved the existence of an $(2n+1)$-order evolution equation, generalizing the Camassa-Holm equation :

$$\sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{a_k}\big(\mathbf{u^{(2k)}}\big)_{\mathbf{t}} + \mathbf{3a_0uu_x} + \sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{a_k}\big(\mathbf{uu^{(2k+1)}} + \mathbf{u_x u^{(2k)}}\big) = \mathbf{0}.$$

Each one of them admits peakon solution; the corresponding N-peakon solution has a waves interaction system of Calogero type. They are all completely integrable in the sense of Arnold-Liouville and they have a common Lax representation.

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