Holomorphic Structures in Vector Bundles Over Complex Surfaces

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1 Introduction

Let X be a compact, connected, complex manifold of dimension two, called *complex surface* or, shortly, *surface*. Let $r \geq 2$ be an integer and let $c_1 \in H^2(X,\mathbb{Z})$, $c_2 \in H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ be fixed cohomology classes. The topological classification of complex vector bundles was given by Wu ([33]): The isomorphism classes of complex topological (smooth) vector bundles are parametrized by the set

$$\{(r, c_1, c_2) \mid r \in \mathbb{Z},\ r \geq 2,\ c_1 \in H^2(X,\mathbb{Z}),\ c_2 \in H^4(X,\mathbb{Z})\}.$$

Two natural problems appear:

I (*existence*) Which complex topological vector bundles over X do admit holomorphic structures ?

II (*classification*) How to parametrize all or parts of the set of holomorphic vector bundles over X ?

In this survey paper we shall present some results on both problems for two very different kind of surfaces: nonalgebraic and ruled surfaces. We mention that a complete answer to the existence problem in the nonalgebraic (surfaces) context is still unknown and, also, that the classification problem is mainly related to the existence of stable vector bundles over surfaces. For more details and proofs, see the quoted papers.

2 Existence of holomorphic structures

Let X be a complex surface. We recall that the set of isomorphism classes of holomorphic line bundles over X forms a group, which is naturally isomorphic to $H^1(X,\mathcal{O}_X^*)$; it is called the *Picard group* and it is denoted by $Pic(X)$. From the exponential exact sequence of sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \overset{exp}{\to} \mathcal{O}_X^* \to 0$$

we get the exact cohomology sequence

$$\to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \overset{\delta}{\to} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X) \to,$$

where $\delta(cls(L)) = c_1(L)$. If we put $Pic_0(X) := Ker\ \delta$, then $Pic(X)/Pic_0(X)$ is isomorphic to a subgroup $Im\ \delta$ of $H^2(X,\mathbb{Z})$, the *Neron-Severi group* of X, denoted by $NS(X)$. Since the topological line bundles are parametrized by the group $H^2(X,\mathbb{Z})$, hence a topological line bundle L has a holomorphic structure if and only if $c_1(L) \in NS(X)$.

Remark. Let E be a topological vector bundle of rank r. If E has a holomorphic structure then $det(E)$ also admits one, hence $c_1(E) = c_1(det(E)) \in NS(X)$. Thus, $c_1(E) \in NS(X)$ is a necessary condition for the existence of holomorphic structures in E.

In the case of algebraic surfaces this condition is also sufficient (cf. [28]):

Theorem 1. (Schwarzenberger) *A topological complex vector bundle E over an algebraic surface X admits a holomorphic structure if and only if $c_1(E) \in NS(X)$.*

This result is no longer true for nonalgebraic surfaces (cf. [19], [15]):

Theorem 2. (Brînzănescu-Flondor) *Let X be a nonalgebraic surface and let $a \in NS(X)$ be fixed. Then, for every holomorphic 2-vector bundle E over X with $c_1(E) = a$, we have*

$$c_2(E) \geq min\ \{a^2/4,\ 2\chi(\mathcal{O}_X) + (c_1(X).a + a^2)/2\ \}.$$

Corollary 3. (Elencwajg-Forster) *Let X be a nonalgebraic surface. Then, there exist topological complex 2-vector bundles E over X with holomorphic structures on $det(E)$, but which do not admit any holomorphic structure.*

More generally, let E be a holomorphic vector bundle over a surface X of rank $r \geq 2$, with Chern classes $c_1(E),\ c_2(E)$. The *discriminant* $\Delta(E)$ is defined by

$$\Delta(E) :=\ c_2(E) - \frac{r-1}{2r}c_1^2(E)\ ;$$

(sometimes the normalization $\Delta_1(E) := \frac{1}{r}\Delta(E)$ is used).

The following necessary conditions are known (cf. [7], [11], [32]):

Theorem 4. (Bănică-Le Potier) *Let X be a nonalgebraic surface and let E be a holomorphic vector bundle over X of rank $r \geq 2$, with Chern classes $c_1(E)$ and $c_2(E)$. Then $\Delta(E) \geq 0$.*

Theorem 5. (Vuletescu) *Let X be a nonalgebraic surface and let E be a holomorphic vector bundle over X of rank $r \geq 2$, with Chern classes $c_1(E)$ and $c_2(E)$. Then $\chi(E) \leq r\chi(\mathcal{O}_X)$.*

Remark. The inequality $\chi(E) \leq r\chi(\mathcal{O}_X)$ is equivalent to the inequality $\Delta(E) \geq 0$ for minimal surfaces, but it is stronger for non-minimal surfaces.

Corollary 6. *Let X be a nonalgebraic surface. Then, there exist topological complex vector bundles E of any rank $r \geq 2$ over X with holomorphic structure on $det(E)$, but which do not admit any holomorphic structure.*

A holomorphic vector bundle E of rank r over X is called *filtrable* if there exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_r = E\ ,$$

where $\mathcal{F}_k$ is a coherent subsheaf of rank k.

Remark. Every holomorphic (algebraic) vector bundle over a projective manifold is filtrable, but on nonalgebraic manifolds there exist holomorphic vector bundles which are not filtrable (cf. [19], [7]).

For a nonalgebraic surface X, $a \in NS(X)$ and r a positive integer we can define the following rational positive number:

$$m(r,a) := -\frac{1}{2r}\ max\ \{\sum_1^r (\frac{a}{r} - \mu_i)^2 \ \mid \mu_1, ..., \mu_r \in NS(X)\ ;\ \sum_1^r \mu_i = a\ \}\ ;$$

(cf. [15], [7]). When X is a 2-torus and $r = 2$ an explicit description of the invariants $m(2, a)$ is given in [15].

A complete answer to the existence problem for filtrable holomorphic vector bundles over nonalgebraic surfaces was obtained by Bănică- Le Potier [7] (the case of rank 2-vector bundles was solved earlier for K3-surfaces by Le Potier [24] and for 2-tori by Brînzănescu- Flondor [15]):

Theorem 6. (Bănică-Le Potier) *A rank $r \geq 2$ topological complex vector bundle E over a nonalgebraic surface X admits a filtrable holomorphic structure if and only if*

$$c_1(E) \in NS(X) \ \ and\ \ \Delta_1(E) \geq m(r, c_1(E)),$$

except when X is a K3-surface with $a(X) = 0$, $c_1(E) \in rNS(X)$ and $\Delta(E) = 1/r$. In this excepted case E admits no holomorphic structures.

Let X be a minimal nonalgebraic surface. By the above results, for fixed rank r and fixed first Chern class $c_1 \in NS(X)$ the only unknown situations are for

$$\Delta_1 \in [0, m(r, c_1)).$$

When $m(r, c_1) \neq 0$ this interval is nonempty and the existence problem I reduces to fill this interval with the *candidates*, the nonfiltrable vector bundles. Thus, in order to solve the existence problem for surfaces we can ask to develop a technique to study nonfiltrable (and even irreducible vector bundles which do admit only trivial subsheaves). We mention that the Serre's method of extensions (see, for example [25]) gives only filtrable vector bundles. There is some progress in this direction for some elliptic surfaces.

A surface is *elliptic* if it admits an elliptic fibration, i.e. a connected holomorphic map $f : X \to S$ from X to a nonsingular curve S, such that the general fibre X_s, $s \in S$, is a smooth elliptic curve. Let $f : X \to S$ be an elliptic surface. We shall assume that f is relatively minimal, i.e. all fibres are free of (-1)–curves. Let $F = \sum n_i D_i$ be a singular fibre of f, where D_i's are the irreducible reduced components and the n_i's are their multiplicities. Let m denotes the greatest common divisor of the n_i's. If $m \geq 2$, then the fibre F is called *multiple fibre* of multiplicity m and we write $F = mD$, where $D = \sum(n_i/m)D_i$ (see [6]).

An elliptic surface $f : X \to S$ is called a *quasibundle* if all smooth fibres are pairwise isomorphic, and the only singular fibres are multiple of smooth (elliptic) curves. If moreover, f has no singular fibres then $f : X \to S$ is said to be a fibre bundle. By an *elliptic bundle* we mean a principal fibre bundle (i.e. the typical fibre and structure group are an elliptic curve T).

The idea is that a better description of the Neron-Severi group could help in solving the existence problem I ! One knows that a nonkählerian elliptic surface $f : X \to S$ is a quasibundle (see [23], or [12], [13]). For nonkählerian surfaces we have the following result (see [9], [12], [13]):

Theorem 7. (Brînzănescu) *Let $f : X \to S$ be the elliptic fibration of a nonkählerian elliptic surface X and let T be the (general) elliptic fibre of f. Then the group $NS(X)/$ Tors $NS(X)$ is isomorphic to the group $Hom(J_S, T)$ of homomorphisms of abelian varieties, where J_S is the Jacobian variety of the curve S.*

Remark. This theorem is an "Appell–Humbert" type result like in the case of complex tori $X = \mathbb{C}^n/\Gamma$ (Γ is a lattice in $\mathbb{C}^n$).

A *primary Kodaira surface* X is an elliptic bundle $f : X \to S$, where S is also an elliptic curve. By using this result and the classical Appell–Humbert Theorem, M. Toma ([30], [31]) obtained the following:

Theorem 8. (Toma) *Any topological complex vector bundle E over a complex torus of dimension 2 or over a primary Kodaira surface, such that $c_1(E) \in NS(X)$ and with $\Delta(E)$ a positive integer, admits a holomorphic structure.*

3 Existence of stable holomorphic bundles

Let X be a projective complex manifold of dimension n, polarized by a very ample divisor (line bundle) H. Let $\mathcal{E}$ be a torsion-free coherent sheaf over X. The number $c_1(\mathcal{E}).c_1(H)^{n-1}$ is called the *degree* of $\mathcal{E}$ *relative* to H and it is denoted by $deg_H(\mathcal{E})$. The number $deg_H(\mathcal{E})/r$, where $r = rank(\mathcal{E})$, is called the *slope* of $\mathcal{E}$ *relative* to H and it is denoted by $\mu_H(\mathcal{E})$. Following Mumford and Takemoto we call an algebraic vector bundle E over X *H-stable* (*semistable*) if for every coherent subsheaf $\mathcal{F} \subset E$ with $0 < rk(\mathcal{F}) < rk(E)$, we have $\mu_H(\mathcal{F}) < \mu_H(E)$ ($\leq$).

Let X be a compact Kähler manifold of dimension n with Kähler metric g and let ω be its Kähler form; it is a real positive closed (1,1)-form on X. Let $\mathcal{E}$ be a torsion-free coherent sheaf over X and let $c_1(\mathcal{E})$ be its first Chern class; it is represented by a real closed (1,1)-form on X. We can define the *degree* of $\mathcal{E}$ *relative* to g by

$$deg_g(\mathcal{E}) := \int_X c_1(\mathcal{E}) \wedge \omega^{n-1} \ ,$$

then the slope, the g-stability, etc.

More generally, let X be a compact complex hermitian manifold of dimension n. A hermitian metric g on X is called a *Gauduchon metric* if its "Kähler form" ω_g satisfies the following second order differential equation:

$$\partial\overline{\partial}\omega_g^{n-1} = 0.$$

The importance of such metrics is given by the following result ([20]):

Theorem 9. (Gauduchon) *Any hermitian metric on X is conformally equivalent to a metric g with $\partial\overline{\partial}\omega_g^{n-1} = 0$. If $n \geq 2$, then this metric is unique up to a positive factor.*

Let g be a Gauduchon metric on X. By an idea of Hitchin, if L is a holomorphic line bundle over X, the *degree* of L *with respect to* ω_g can be defined by

$$deg_g(L) := \int_X \frac{i}{2\pi} F \wedge \omega_g^{n-1} \ ,$$

where F is the curvature of any hermitian connection on L, compatible with $\overline{\partial}_L$. Since any two such forms differ by a $\partial\overline{\partial}-$exact form, $deg_g(L)$ is independent of the choice of the connection. Having defined the degree of holomorphic line bundles, the degree of a torsion-free coherent sheaf $\mathcal{E}$ is defined by $deg_g(\mathcal{E}) := deg_g(det(\mathcal{E}))$ and the definition of the $g-$stability can be repeated verbatim.

Remark. Let X be a nonalgebraic surface (even non-Kähler). Let E be an irreducible holomorphic vector bundle (examples were constructed by Elencwajg-Forster [19], Bănică-Le Potier [7], Toma [30]). Then, by definition, E is $g-$stable with respect to any Gauduchon metric g (i.e. E is *absolutely stable*).

Let X be a complex surface, $r \geq 2$ an integer, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z})$. If (X, H) is an algebraic surface, we denote by $\mathcal{M}_H(r; c_1, c_2)$ the set (moduli space) of isomorphism classes of $H-$stable vector bundles of rank r, with Chern classes c_1 and c_2. If (X, ω) (resp. (X, g)) is a Kähler surface (resp. a non-Kähler surface) we denote by $\mathcal{M}_\omega(r; c_1, c_2)$ (resp. $\mathcal{M}_g(r; c_1, c_2)$) the corresponding moduli spaces. The study of these moduli spaces represents solution of the Problem II and, a first question is "when is $\mathcal{M}_H$ (resp. $\mathcal{M}_\omega$, $\mathcal{M}_g$) nonempty ?".

Remark. In the case X nonalgebraic, and for $rk(E) = 2$ and $\Delta_1 \in [0, m(2, c_1))$, any holomorphic 2-vector bundle E with $c_1(E) = c_1,\ \Delta_1(E) = \Delta_1$ would be irreducible (since it is nonfiltrable) and hence $g-$stable for any Gauduchon metric g. Thus, the solution of Problem I gives in this case also the solution of the above question.

Let X be an algebraic surface. We have the following *asimptotic* result ([22]):

Theorem 10. (Hirschowitz-Laszlo) *Let X be an algebraic surface over $\mathbb{C}$ and let H be a fixed very ample line bundle over X. For any integer $r \geq 2$ and any $c_1 \in NS(X)$ there exists an integer $N(X, H, r, c_1)$, such that for all $\ell \geq N$ there exists a $H-$stable vector bundle of rank r with $c_1(E) = c_1$ and $c_2(E) = \ell$.*

Now, we restrict ourselves to the case of ruled surfaces and rank-2 vector bundles, but the results will be almost complete (i.e. not asimptotic).

Let C be a curve of genus g and let $\pi : X \to C$ be a ruled surface. We shall denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the divisor p_0 on C ($p_0 \in C$ fixed, $f_0 = \pi^{-1}(p_0)$). Any element of $Num(X) = NS(X) \cong H^2(X, \mathbb{Z})$ can be written $aC_0 + bf_0$ with $a,\ b \in \mathbb{Z}$ and C_0 a minimal section ($C_0^2 = -e$); see, for example [21]. If $L \in Pic(C)$ we shall write $L = \mathcal{O}_C(k) \otimes L_0$, where $L_0 \in Pic_0(C)$ and $k = deg(L)$. We shall also denote by $F(aC_0 + bf_0) = F \otimes \mathcal{O}_X(aC_0) \otimes \pi^*(\mathcal{O}_C(b))$ for any sheaf F over X and any $a,\ b \in \mathbb{Z}$.

Let E be an algebraic rank 2-vector bundle over X with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z}^2$, and $c_2 = \gamma \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$. Since the fibres of π are isomorphic to $\mathbb{P}^1$, we can speak about the generic splitting type of E and we have $E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$, for a general fibre of π, where $d \geq d'$, $d + d' = \alpha$. The integer d is a *first numerical invariant* of E. The *second numerical invariant* r is obtained by the following normalization:

$$-r = inf\{\ell \mid \exists\, L \in Pic(C),\ deg(L) = \ell,\ \ H^0(X, E(-dC_0) \otimes \pi^*(L) \neq \{0\}\ \}$$

(i.e. $r =$ the greatest degree of a rank 1-subsheaf of $\pi_*(E(-dC_0)))$. Let $M(c_1, c_2, d, r)$ be the set of isomorphism classes of 2-vector bundles over X with fixed invariants $c_1,\ c_2,\ d$ and r.

Remark. The moduli spaces $M(c_1, c_2, d, r)$ are defined independently of any ample divisor (line bundle) over X.

For the following result, see [16], [17]:

Theorem 11. (Brînzănescu-Stoia) *For every vector bundle E in the set $M(c_1, c_2, d, r)$ there exist $L_1,\ L_2 \in Pic_0(C)$ and $Y \subset X$ a locally complete intersection of codimension 2 in X, or the empty set, such that E is given by an extension*

$$(*)\ \ 0 \to \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \to E \to \mathcal{O}_X(d'C_0 + sf_0) \otimes \pi^* L_1 \otimes \mathcal{I}_Y \to 0\ ,$$

where $c_1 = (\alpha, \beta) \in \mathbb{Z}^2$, $c_2 = \gamma \in \mathbb{Z}$, $d + d' = \alpha$, $r + s = \beta$, and $\ell(c_1, c_2, d, r) := \gamma + \alpha(de - r) - \beta d + 2dr - d^2 e = deg(Y) \geq 0$.

We have (see [8]):

Theorem 12. (Brînzănescu) *The set $M(c_1, c_2, d, r)$ is bounded.*

The problem of the nonemptiness of the moduli spaces $M(c_1, c_2, d, r)$ is solved in [1], [3] :

Theorem 13. (Aprodu-Brînzănescu) *The moduli space $M(c_1, c_2, d, r)$ is nonempty if and only if $\ell := \ell(c_1, c_2, d, r) \geq 0$ and one of the following conditions holds:*

(i) $2d > \alpha$, or

(ii) $2d = \alpha$, $\beta - 2r \leq g + \ell$.

For any algebraic surface X, let $\mathcal{C}_X$ be the ample cone in $Num(X) \otimes \mathbb{R}$ generated by ample divisors. We fix the Chern classes $\tilde{c}_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. We shall use (see, for example, [26], [27]) the following definitions:

Definition. (Qin) (i) For $\zeta \in Num(X) \otimes \mathbb{R}$, we define

$$W^\zeta := \mathcal{C}_X \cap \{x \in Num(X) \otimes \mathbb{R} \mid x.\zeta = 0\}.$$

(ii) We define $\mathcal{W}(\tilde{c}_1, c_2)$ to be the union of W^ζ, where ζ is the numerical equivalence class of $(2F - \tilde{c}_1)$ for some divisor F, and which satisfies the conditions:

$$-(4c_2 - c_1^2) \leq \zeta^2 < 0\ .$$

(iii) An *wall* of type $(\tilde{c}_1, c_2)$ is an element W^ζ, where ζ satisfies the conditions in (ii). A *chamber* of type $(\tilde{c}_1, c_2)$ is a connected component of the set $\mathcal{C}_X \setminus \mathcal{W}(\tilde{c}_1, c_2)$.
(iv) A numerical equivalence class ζ which represents a nonempty wall W^ζ is normalized, in the case of a ruled surface X, if the integer $(\zeta.f)$ is positive.

(v) Let W^ζ be a nonempty wall of type $(\tilde{c}_1, c_2)$ and let $\ell_\zeta(\tilde{c}_1, c_2)$ be the integer $c_2 + (\zeta^2 - \tilde{c}_1^2)/4$. We define $E_\zeta(\tilde{c}_1, c_2)$ to be the set of isomorphism classes of 2-vector bundles E given by nontrivial extensions

$$0 \to \mathcal{O}_X(F) \to E \to \mathcal{O}_X(\tilde{c}_1 - F) \otimes \mathcal{I}_Y \to 0 ,$$

where F is a divisor such that ζ is the numerical equivalence class of $(2F - \tilde{c}_1)$, and where $Y \subset X$ is a locally complete intersection of codimension 2 in X such that $deg(Y) = \ell_\zeta(\tilde{c}_1, c_2)$.

Remark. The definitions (i)-(iv) depend only on the numerical type (c_1, c_2), where c_1 is the numerical equivalence class of $\tilde{c}_1$. We fix the numerical Chern classes $c_1 = (\alpha, \beta) \in Num(X)$, $c_2 \in \mathbb{Z}$ and integers d, r such that $2d > \alpha$, $\ell(c_1, c_2, d, r) \geq 0$. We denote by $\zeta = (d - d')C_0 + (r - s)f_0$ and we have that the condition $\ell(c_1, c_2, d, r) \geq 0$ is equivalent to the condition $-(4c_2 - c_1^2) \leq \zeta^2$, and that $\zeta + c_1$ is the numerical equivalence class of $2F$ for F a divisor on X. If we suppose, moreover, that $\zeta^2 < 0$ and that there exists an ample line bundle L over X such that $c_1(L).\zeta = 0$, then the element ζ represents a nonempty wall of (numerical) type (c_1, c_2) and we have $E_\zeta(\tilde{c}_1, c_2) \subset M(c_1, c_2, d, r)$ (see [3]).

We shall investigate the stability of vector bundles in the moduli spaces $M(c_1, c_2, d, r)$ (see [2], [4]):

Theorem 14. (Aprodu-Brînzănescu) *Let $E \in M(c_1, c_2, d, r)$. Then, there exists an ample line bundle H such that E is $H-$stable if and only if $2r - \beta < min\{0, e(2d - \alpha)/2\}$ and the extension (*) of E is nonsplitting.*

As consequences we get the following results (see, also [29]):

Corollary 15. (Takemoto-Bogomolov) *Let X be a ruled surface and let E be an algebraic 2-vector bundle over X with $c_1(E) = c_1$, $c_2(E) = c_2$. If E is $H-$stable for some ample line bundle H, then $\Delta(E) \geq 0$.*

Remark. Then, if $4c_2 - c_1^2 < 0$, we get $\mathcal{M}_H(c_1, c_2) = \emptyset$, for all ample line bundles H.

Corollary 16. (Takemoto) *Let H be an ample line bundle over a ruled surface X. An algebraic 2-vector bundle E over X with $\Delta(E) = 0$ is $H-$stable if and only if there exist a stable 2-vector bundle F over the curve C and a line bundle L over X such that $E \cong \pi^*(F) \otimes L$.*

We shall assume now that $4c_2 - c_1^2 > 0$. Let $H \equiv aC_0 + bf_0$ be an ample divisor. Then $a > 0$ and $b > ae$ if $e \geq 0$ and $a > 0$ and $b > ae/2$ if $e < 0$. Therefore, for a ruled surface, the ample cone has a simple description. Moreover, from definition (iii), we get that there exist a finite number of walls and chambers. Denote by $\mathcal{C}_{f_0}$ the chamber of type (c_1, c_2) such that the $[f_0]-$axis in $Num(X) \otimes \mathbb{R}$ is part of the boundary of $\mathcal{C}_{f_0}$. Firstly, suppose that the ample line bundle H belongs to some chamber. It is well-known that if H_1 and H_2 lie in the same chamber then, the moduli spaces $\mathcal{M}_{H_1}(c_1, c_2)$ and $\mathcal{M}_{H_2}(c_1, c_2)$ can be naturally identified. We have (see [2], [4]):

Corollary 17. (Aprodu-Brînzănescu) *Let X be a ruled surface. Assume that X is not $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{C}$ be any chamber of type (c_1, c_2) different from $\mathcal{C}_{f_0}$. Then $\mathcal{M}_{\mathcal{C}}(c_1, c_2)$ is nonempty.*

Corollary 18. (Aprodu-Brînzănescu) *Let X be a ruled surface. Then $\mathcal{M}_{\mathcal{C}_{f_0}}(c_1, c_2)$ is nonempty if and only if α is even and the intersection*

$$[\beta/2 - (g + c_2 - c_1^2/4)/2,\ \beta/2) \cap \mathbb{Z}$$

is nonempty.

Remark. By applying the above two corollaries the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ is also settled.

Now, suppose that H lies on some nonempty wall. Here, we obtain only a partial result (see [2], [4]):

Corollary 19. (Aprodu-Brînzănescu) *Let X be a ruled surface different from $\mathbb{P}^1 \times \mathbb{P}^1$ with $e \geq 0$ and assume $g \leq e + 1$. Let $H \equiv aC_0 + bf_0$ be an ample line bundle lying on some nonempty wall W of type (c_1, c_2) and denote $b/a = k$. Assume either $\zeta.f_0 \geq 2$ for all normalized numerical equivalence classes ζ which represent the wall W or $4c_2 - c_1^2 > 2k - e$. Then $\mathcal{M}_H(c_1, c_2)$ is nonempty.*

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