Differential Spaces and Singularities of Space-Time

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Abstract

First, a version of differential spaces is defined, which can be used to describe the singularities in General Relativity. Then this concept is applied to the most common singularities: In the Friedmann space-times, all freely falling pointlike particles can pass the final collapse and so emerge from the Big Bang. The topology of this singularity is very natural and agrees with the topology induced from the Fronsdal embedding. It is possible to glue together the Black Hole and the White Source of the maximal analytic extension of Schwarzschild space-times. In this way, some of the classical problems with this extension are solved. Finally, an example is given, where the mass of a particle is changed while passing through a singularity.

1 Introduction and Summary

The well-known theorems of Hawking and Penrose state that space-times form singularities under very general conditions. As by definition singularities are no points on differentiable manifolds, the first results in this field have been obtained by limiting processes from regular points. But recently, some generalizations of differentiable manifolds have been developed so far that they can be applied to General Relativity. The important fact is that a consistent mathematical formalism has been found such that now, the singular points can be considered as points of space-time. The details of this formalism are not important; all results described in this paper only need a suitable topology which agrees with the standard topology on all regular points, and some basic differential calculus.

This paper first describes such a generalization of differential manifolds which has been useful in practical calculations. Then, in sections 3 and 4, some results on Friedmann and Schwarzschild singularities are reviewed. It is shown that all freely falling pointlike particles can pass the final collapse of our universe and so emerge from the Big Bang of the next universe.

The topology constructed for this singularity is very natural and agrees with the topology induced from the Fronsdal embedding.

The maximal analytic extension of the exterior Schwarzschild space-time contains a black hole and a white source. So there are regions, where causality is violated. But using the formalism of d-spaces described below, it is possible to glue the black hole to the white source in such a way that causality is conserved locally (but not globally): A pointlike particle falling into the black hole from region I leaves the white source and finally reaches region III.

All these results have been published in recent papers. But a completely new phenomen is discussed in section 5. If space-time is modelled by two intersecting hyperplanes in $\mathbb{R}^5$, Einstein's equations for the vacuum are satisfied even in their intersection – at least in some formal sense.

A particle travelling from one hyperplane to the other on a geodesic, changes its mass. A light like particle can change even to a spacelike one and vice versa.

These results are true in classical physics. But one expects that physics in and near a singularity is dominated by quantum effects. There are arguments, [21], however, that the calculations of the present paper are approximately valid also in the quantum regime:
The transition amplitude of a particle to travel from a point x to a point y is

$$\int \exp\,(i\int L(\gamma)d\tau)d\gamma = \int \exp\,(\frac{i}{\hbar}\,m\int\sqrt{g_{jk}\,u^j\,u^k}\,d\tau)\,d\gamma\;, \tag{1}$$

where L is the Lagrangian which depends on the path γ from x to y. The u^j are the components of the tangent vector $\frac{d\gamma}{d\tau}$, and m is a constant. (Remember that the 4-momentum P of a particle is proportional to the 4-velocity $\frac{d\gamma}{d\tau}$.) By definition, the geodesics are the curves, for which the integral

$$\int\sqrt{g_{jk}\,u^j\,u^k}\,d\tau$$

is stationary. In addition, for geodesics in Friedmann or Schwarzschild space-times, it is finite even in the limit $\eta \to 0$ respectively $r \to 0$. This means that the exponent in (1) varies only little. Therefore there is not much destructive interference in the "neighbourhood" of a geodesic $\gamma = \gamma_0$, and we expect that the main contribution to the transition amplitude (1) comes from the geodesic γ_0 and the neighbouring paths.

2 d-Spaces

The characteristic property of a manifold M is that it is locally homeomorphic to $\mathbb{R}^n$. This fact may be expressed by the local coordinates, i.e. by local functions $M \longrightarrow \mathbb{R}$. Similarly, by definition, a C^r-manifold admits C^r-functions $M \longrightarrow \mathbb{R}$, but not $C^{r+1}-$functions.

This concept has been generalized by the pair of a topological space M and a set $\mathfrak{C}$ of continuous local functions $M \longrightarrow \mathbb{R}$. Of course, one wants to add and multiply these functions. So it is natural to require that $\mathfrak{C}$ is at least an algebra:

Definition 1 *Let M be a topological space. The pair $(M, \mathfrak{C})$ is called* **differential space** *(or* **d-space***), and $\mathfrak{C}$* **differential structure***, if $\mathfrak{C}$ is a sheaf of continuous real-valued functions on M which form an algebra (w.r.t. pointwise operation).*

This definition has first been given by Gerstner [15], [16]. It generalizes Sikorski's definition [27] which requires global functions. In addition, the composition of a function $g \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$ with n arbitrary functions of $\mathfrak{C}$ must again be a function of $\mathfrak{C}$, and M must carry the initial topology of $\mathfrak{C}$. The latter axiom is one of the reasons, why Sikorski's definition is not able to describe the singularities of General Relativity properly [24]. Therefore Heller and Sasin proposed [25] to use Mostow spaces [26], which were later called structured spaces. They are slightly more special than the d-spaces of Definition 1.

Of course, there are many other successful generalizations of differential manifolds. We only mention the work of Barthel [2], Fröhlicher and Kriegl [12], Gähler [13] and Spallek [28].

On d-spaces, differentiation, differential forms and the exterior derivative are introduced [15], [16] in a way similar to Sikorski's differential spaces. But tangent vectors need to be defined such that the dimesion of the tangent space does not depend on the differentiability class of the functions in $\mathfrak{C}$ (which was the case in the older theories, cf [5]):

Definition 2 *Let $(M, \mathfrak{C})$ be a d-space, $x \in M$ and $\mathfrak{C}_x$ the stalk at x. A map*

$$V: \quad \mathfrak{C}_x \to \mathbb{R}$$

is called **tangent vector** *to $(M, \mathfrak{C})$ at x, if for all $n \in \mathbb{N}$, all $f_1 \ldots, f_n \in \mathfrak{C}_x$ and all germs $\alpha \in C^1(\mathbb{R}^n, \mathbb{R})$ at $y := (f_1(x), \ldots, f_n(x)) \in \mathbb{R}^n$, the equation*

$$V(\alpha \circ (f_1(x), \ldots, f_n(x))) = \sum_{i=1}^{n} (\partial_i \alpha) \cdot V(f_i)$$

holds, provided $\alpha \circ (f_1, \ldots, f_n) \in \mathfrak{C}_x$. Here ∂_i denotes the partial derivative of α w.r.t. the i-th argument.
The vector space formed by the tangent vectors to $(M, \mathfrak{C})$ at x is called **tangent space** *T_xM.*

The dimension of T_xM can easily be computed:
We call a subset F of the stalk $\mathfrak{C}_x$ **dependent**, if there is a germ $g \in F$ and a finite subset $\{f_1, \ldots, f_n\} \subseteq F$ not containing g such that

$$g = \alpha \circ (f_1, \ldots, f_n)$$

holds for some α of class C^1. Choose an independent subset $G \subseteq \mathfrak{C}_x$ such that every $f \in \mathfrak{C}_x$ can be written as $f = \alpha \circ (f_1, \ldots, f_n)$ with suitable $(f_1, \ldots, f_n) \in G$; $n \in \mathbb{N}$ and α of class C^1.
Write G as $G = \{f_j\}_{j \in J}$. Then Definition 2 shows that $\{V_j\}_{j \in J}$ defined by

$$V_j(f_k) = \delta_{jk}$$

is a basis of T_xM.

Note that this kind of theory cannot define a satisfactory integration, because no set of partial funcitons $M - \rightarrow \mathbb{R}$ is defined (see, e.g. [7], [10]).

According to Definition 1, the topology of M may be finer than the initial topology of $\mathfrak{C}$. In many cases, however, it is desirable to define the topology of M by $\mathfrak{C}$. But in contrast to Sikorski's differential spaces, it is not possible here to use simply the initial topology, because deleting some open sets may have the effect of deleting also some local functions which are not restrictions of functions defined on a larger set. So we introduce the following notation:

Definition 3 *Let $(\mathfrak{F}, \gamma)$ be a sheaf, where γ is the topology, w.r.t. which $\mathfrak{F}$ has been defined. A topology α that is coarser than γ, is called a* **slackening** *of $\mathfrak{F}$, if for every $V \in \gamma$ and every $f \in \mathfrak{F}(V)$, there are $U \in \alpha$; $U \supseteq V$ and $g \in \mathfrak{F}(U)$ such that $f = g|_V$ holds.*
According to [15], [16], there exists a minimal slackening μ, such that all functions $g \in \mathfrak{F}(U)$; $U \in \mu$ are continuous. This μ is called **initial topology** *of $(M, \mathfrak{C})$.*

It is not difficult to construct this initial topology explicity (see [15], [16]).

3 The closed Friedmann-Universe

The Standard Model of cosmology uses - at least for part of the time developement - the "Robertson-Walker" metric

$$ds^2 = S^2(\eta)(-d\eta^2 + d\chi^2 + f^2(\chi)(d\vartheta^2 + sin^2\vartheta d\varphi^2)) \ .$$

Here, we are interested in the case, where the curvature of the space-like hypersurfaces $\eta = const$ is positive. In particular, we assume that the energymomentum tensor is dominated by incoherent radiation, which was probably the case in the very early time after the Big Bang. This leads to

$$\chi \in [0, \pi], \quad \vartheta \in [0, \pi], \quad \varphi \in [0, 2\pi),$$

$$S(\eta) = a \sin\eta, \quad t = a(1 - \cos\eta), \quad \frac{dt}{d\eta} = S(\eta), \quad a \in \mathbb{R}.$$

This metric is clearly singular at the time $t = 0$ ($\Rightarrow \eta = 0$), the "Big Bang". For $\eta = \pi$, there is an other singularity, the final collapse. These "points" do not belong to any differential manifold, but to a d-space. We shall discuss some of their properties, which do not depend on the special version of the d-space used. The important fact is that these singularities belong to space-time and are boundary points of differential manifolds, so that continuity arguments may be applied.

One usually assumes that the universe reexpands after the final collapse, i.e. the final collapse is the Big Bang of a new universe. Sometimes, it is argued that all information is lost in such a singularity. Consequently it may be possible that the new universe is not of the type discussed here, but expands forever and has no final collapse. Heller and Sasin [23] have pointed out that this is not possible within the theory of d-spaces, because (in the simplest cosmological models) the behaviour of the universe is determined by the initial condition of the solution to Friedmann's equation. If the universe is described by a single d-space M before, at and after the Big Bang, this integration constant must be the same for the whole solution at all times.(Cf. also Buchner, Büschel [6].)

The next problem is the topology at the singularity. There has been considerable controversy about this point (see, e.g. [3], [8], [9], [14], [18], [20]). A simple answer to this question has been proposed by Abdel-Megied et al. in [1]: The differential structure of the d-space M describing space-time, is formed by the normal coordinates at all non-singular points $x \in M$. The topology of M is then determined by Definition 3. In the present case it is simply the initial topology. This means that singularities which cannot be reached on geodesics [14], consist of discrete points.

In order to compute this topology, we define

$$(x^1, x^2, x^3, x^4) = (\chi, \vartheta, \varphi, t),$$

and assume that $x^1 \neq 0, \pi,\ x^2 \neq 0, \pi,$ and $x^3 \neq 0$, which is always possible due to the spherical symmetry. For the same reason, the geodesics may be chosen such that their tangent vectors $u^j = \frac{dx^j}{d\tau}$ (w.r.t. an affine parameter τ) satisfy $u^2 = 0$ and, for $u^3 \neq 0$ also $\vartheta = \frac{\pi}{2}$. Then the geodesic equations may be integrated to give

$$\begin{aligned} (u^1)^2 &= \frac{1}{S^4}\,(C_2^2 - \frac{C_1^2}{f^2})\ , \\ u^3 &= \frac{C_1}{S^2 f^2}\ , \\ u^4 &= \pm\sqrt{\frac{C_2^2}{S^2} - k}\ , \\ \frac{d\tau}{dt} = \frac{1}{u^4}\ ,& \quad \frac{d\tau}{d\eta} = \frac{S}{u^4}\ . \end{aligned} \tag{2}$$

Here C_1 and C_2 are constants. $k = -1,\ 0,$ or 1 for timelike, null, and spacelike geodesics, respectively. The coordinate intervals from the Big Bang to an arbitrary point on these geodesics are finite:

$$\Delta\chi = \int_0^{\bar{\tau}} u^1 d\tau = \int_{\eta_0}^{0} u^1 \frac{S}{u^4}\, d\eta = \pm \int_{\eta_0}^{0} \frac{1}{f}\, \frac{\sqrt{C_2^2 f^2 - C_1^2}}{\sqrt{C_2^2 - k\, S^2}}\, d\eta\ ,$$

$$\Delta\varphi = \int_{\eta_0}^{0} u^3 \frac{S}{u^4}\, d\eta = \pm \int_{\eta_0}^{0} \frac{C_1}{f^2}\, \frac{d\eta}{\sqrt{C_2^2 - k\, S^2}}\ .$$

Therefore the topology of M is generated [1] by the open neighbourhoods of the non-singular points (in conventional space-time) and by the neighbourhood basis

$$\{(x^1, x^2, x^3, x^4) \mid |x^j - x_o^j| < \varepsilon;\ j = 1, 2, 3; -\varepsilon < x^4 < \varepsilon\}$$

respectively

$$\{(x^1, x^2, x^3, x^4) \mid |x^j - x_o^j| < \varepsilon;\ j = 1, 2, 3;\ \pi - \varepsilon < x^4 < \pi + \varepsilon\}$$

of the singular points x_0. (Notice that τ is a well defined function of η; see [1]).

(2) leads to an other interesting result: In the limit $\eta \to 0; \quad \eta \to \pi$, all geodesics are well defined, although the velocities become infinite. The fact that in the singularity, the four-velocities are infinite, merely reflects the limit $\lim_{\eta \to 0} S(\eta) = 0$, i.e. the proper time is not a good parameter at $\eta = 0$.

This means that all pointlike particles can pass through the final collapse to the new universe [1]. Here, the restriction to pointlike particles is essential, because some of the tidal forces become infinite. Of course, the various interactions are so strong immediately before and after this singularity, that the particles may be continuously transformed to others. So there is little chance to observe "debris from the universe before ours".

4 The Exterior Schwarzschild Space-Time

The line element of a spherically symmetric empty space-time is

$$(3) \qquad ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1}\, dr^2 + r^2\, (d\,\vartheta^2 + \sin^2 \vartheta\, d\varphi^2)\ .$$

As is well known, the singularity of $r = 2M$ is only a coordinate singularity. There are geodesics that reach it at a finite affine parameter and can be continued to the region $r < 2M$. There is a larger space called the "maximal analytic extension" of (3) with the set $r = 2M$ as a regular subset. It contains four different regions:

$$I:\ u > |v|, \quad II: v > |u|, \quad III:\ u < |v|, \quad IV: v < |u|.$$

$$(4) \qquad (I) \quad \begin{cases} u &= (r/2M-1)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \cosh(t/4M), \\ v &= (r/2M-1)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \sinh(t/4M), \end{cases}$$

$$(5) \qquad (II) \quad \begin{cases} u &= (1-r/2M)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \sinh(t/4M), \\ v &= (1-r/2M)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \cosh(t/4M), \end{cases}$$

$$(6) \qquad (III) \quad \begin{cases} u &= -(r/2M-1)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \cosh(t/4M), \\ v &= -(r/2M-1)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \sinh(t/4M), \end{cases}$$

$$(7) \qquad (IV) \quad \begin{cases} u &= -(1-r/2M)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \sinh(t/4M), \\ v &= -(1-r/2M)^{\frac{1}{2}}\, e^{\frac{r}{4M}}\, \cosh(t/4M). \end{cases}$$

The metric

$$(8) \qquad ds^2 = \frac{32M^3}{r} e^{\frac{-r}{2M}} (du^2 - dv^2) + r^2\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

is valid in the entire space-time. One immediately realizes that $r = 2M$ is no longer a singularity of (8). But the singularity $r = 0$ now has two unconnected parts defined by the hyperbola $v^2 - u^2 = 1$. Its points with $v > 0$ form a "black hole", those with $v < 0$ a "white source". Of course, such a white source violates causality, because timelike and lightlike geodesics begin there "without any reason". Therefore it makes sense trying to glue the two unconnected parts of the singularity together. We will show that this is in fact possible, if the points (u, v) on the hyperbola are identified with $(-u, -u)$ [22].

Instead of discussing the proper and improper limits of the metric for $r \to 0$, it is easier to compute a sufficient number of geodesics ending respectively beginning in the singularity [21]. This is so, because they are well defined in $r = 0$, and because in classical differential geometry, the Christoffel symbols can be deduced from the geodesics.

The first integrals of the geodesics are

$$(9) \qquad \begin{aligned} (\tfrac{dr}{dt})^2 &= E^2 + (1 - \tfrac{2M}{r})\,(k - \tfrac{L^2}{r^2}), \\ \tfrac{d\varphi}{d\tau} &= \tfrac{L}{r^2}\,, \quad \tfrac{d\vartheta}{d\tau} =, 0 \\ \left|\tfrac{dt}{d\tau}\right| &= \tfrac{E}{1-\frac{2M}{r}}, \end{aligned}$$

where $k = +1,\ 0,\ -1$ for spacelike, null, and timelike geodesics, respectively. τ is the eigentime and E, L, M are constants. For $(k = -1), E$ is the energy of the particle travelling on the geodesic and L its angular momentum, both in units of its mass. In the derivation of (9), it has been assumed that $\vartheta = \frac{\pi}{2}$ holds on the geodesic, which is always possible because of the spherical symmetry and the fact that $\dfrac{d^2\vartheta}{d\tau^2} = 0$ everywhere for the initial conditions $\vartheta = \frac{\pi}{2}\,,\ \frac{d\vartheta}{d\tau} = 0$.

Notice the modulus of $\frac{dt}{d\tau}$ in (9). It appears, because the first integral $\frac{dt}{d\tau}$ is obtained by an integration of its logarithm. Therefore the sign of $\frac{dt}{d\tau}$ is undetermined. (This fact has been missed in [22]).
(9) are most easily integrated in the case of radially falling photons (i.e. $k = L = 0$). The limit $r \to 0$ is well defined. The geodesic could in principle be continued to negative values of r.
If one wants to identify the points (u, v) and $(-u, -v)$ on the singularity, one needs continuous first, second and third derivatives of the geodesics, written in the coordinates $(u, v, \vartheta, \varphi)$. This can be achieved in the present case ($k = L = 0$), if the signs of $\frac{dr}{d\tau}$ and $\frac{dt}{d\tau}$ change at $r = 0$. [See (5), (7): $\frac{du}{dr}$ and $\frac{dv}{dr}$ reverse their sign, if one passes from a photon absorbed in the black hole to a photon emitted from the white source].

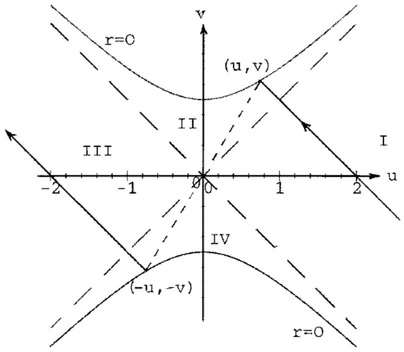

Figure: A photon (straight line in the (u/v)-plane) coming from region I is absorbed in the black hole (region II) in a point with coordinates (u,v,ϑ,ϕ) and emitted again from the white source in a point with the coordinates$(-u,-v,\vartheta,\phi)$ in region IV. It escapes to region III.

For $k \neq 0$ or $L \neq 0, \tau$ is no longer a good parameter for the geodesics in $r = 0$. But for other parameters ρ (e.g. $\rho = \sqrt{r}$), everything is locally well defined. Now the metric (8) is invariant under the transformation $(u, v) \mapsto (-u, -v)$ from region II to region IV, as can be seen from the inverse equations to (4)-(7):

$$(10) \qquad \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = u^2 - v^2 \, ,$$

$$(11) \qquad t = \begin{cases} 4M tanh^{-1}\frac{u}{v} & \text{in I and III} \, , \\ 4M tanh^{-1}\frac{v}{u} & \text{in II and IV.} \end{cases}$$

This means that also for a general geodesic absorbed by the black hole in the point (u, v), and for an other geodesic emitted by the white source in $(-u, -v)$, all limits $r \to 0$ of the derivatives are equal, provided k and L are the same, and $\frac{dr}{d\rho}$ and $\frac{dt}{d\rho}$ change sign. [See again (5) and (7)]. This has two important consequences: First, the two unconnected parts of the singularity may be identified in the way discussed above. Second, the topology at the singularity is just the topology induced by the coordinates $(u, v, \vartheta, \varphi)$, if it is defined as the initial topology of the normal coodinates at all non-singular points (see section 3).

If the black hole and the white source are identified, a pointlike particle starting in region I and falling into the black hole, finally escapes into region III. And vice versa, a particle starting in region III can reach region I through the singularity. This violates causality, because the particle coming from region I may be scattered in region III such that it falls back into the singularity and enters region I again. In general, it will intersect there its own world line.

This causality violation can be avoided [21], if the black hole is glued to the white source of a different Schwarzschild space-time. In this way, one obtains an infinite series of exterior Schwarzschild solutions.

5 Space-Times with Edges

The closed radiation-filled Friedmann-universe can be embedded into $\mathbb{R}^5$ with the metric

$$ds^2 = -(d\,Z^1)^2 + \sum_{j=2}^{5} (d\,Z^j)^2 \tag{12}$$

by the relations [19]

$$\begin{aligned} Z^1 &= a\,\eta\,, \\ Z^2 &= a\sin\eta\,\cos\chi\,, \\ Z^3 &= a\sin\eta\,\sin\chi\,\cos\varphi\,, \\ Z^4 &= a\sin\eta\,\sin\chi\,\sin\vartheta\,\cos\varphi\,, \\ Z^5 &= a\sin\eta\,\sin\chi\,\sin\vartheta\,\sin\varphi. \end{aligned}$$

Near the Big Bang $\eta = 0$, this hypersurface may be approximated by a spherically symmetric (in the coordinates χ, ϑ, φ) cone. So it is not surprising that all geodesics can be continued through the final collapse respectively the Big Bang.

For the exterior Schwarzschild solution, the Fronsdal embedding [11] into $\mathbb{R}^6$ does not give any interesting results, because the singularity is at $Z^1 = \pm Z^2 = \infty$. But there is a simple example, where the embedding clarifies the physical situation:

In $\mathbb{R}^5$ with the metric (12), we consider the two half hyperplanes

$$\begin{gathered} Z^i = x^i\,, \quad i = 1, \cdots, 4, \\ Z^5 = c\,|ax^1 + bx^4\,|, \\ a,\, b \geq 0\,, \quad c > 0. \end{gathered} \tag{13}$$

Of course, each of these hyperplanes satisfies Einstein's equations of empty space-times. Even in the "edge" $ax^1 + bx^4 = 0$, one can say that Einstein's equations are satisfied in some sense: From

$$\begin{aligned}\frac{\partial Z^\alpha}{\partial x^1} &= \delta_i^\alpha \quad \text{for } \alpha = 1, \dots, 4 \ , \\ \frac{\partial Z^5}{\partial x^1} &= -a\,c + 2\,a\,c\,\vartheta(ax^1 + bx^4) =: a\,c\,T \ , \\ \frac{\partial Z^5}{\partial x^4} &= b\,c\,T \ ,\end{aligned}$$

with

$$\vartheta(x) \ := \ \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x \geq 0, \end{cases}$$

one finds

$$\begin{aligned} g_{ik} = \delta_{ik} \quad \text{for } i \text{ or } k = 2,\ 3 \ , \\ g_{11} = -1 + a^2\,c^2\,T^2, \\ g_{14} = a\,b\,c^2\,T^2, \\ g_{44} = 1 + b^2\,c^2\,T^2 \ . \end{aligned} \tag{14}$$

If the Ricci tensor is computed for this metric, it contains squares of δ-"functions", which are not defined. But these terms cancel, so that the formal expression for the Ricci tensor vanishes. The geodesic equations cannot be computed from (14) in a straight-forward manner, because the Christoffel symbols contain quotients of $\delta(ax^1 + bx^4)$ and T. More serious is the problem that the geodesic equation

$$u^i + \Gamma^i_{jk} u^j u^k = 0$$

contains products of a δ-"function" with the components of u, whose discontinuity is to be determined by this equation. In addition, the physical interpretation of this geodesic equation is questionable. One can avoid these difficulties by looking at the embedding (13). The change of energy and momentum at the edge of space-time is then determined by the requirement that the particle whose 4-momentum is described by the geodesic under discussion, is

I confined to the half planes,

II travelling on straight lines within each half plane,

III experiencing a "force" at the edge which is perpendicular to the half plane, on which it will travel **after** having passed the edge.

Consider a particle starting in the half plane

$$I: \quad ax^1 + bx^4 + \frac{1}{c}\,x^5 = 0$$

and then passing to the half plane

$$II: \quad ax^1 + bx^4 - \frac{1}{c}x^5 = 0$$

[cf. (13)]. The particle's 4-momentum (as a **contra**variant vector) is denoted by

$$P_I = (A_I,\, 0,\, 0,\, B_I,\, C_I)$$

on the half plane I and similarly on II. The contravariant normal vectors to the half planes are

$$\begin{aligned} n_I &= (-a,\, 0,\, 0,\, b,\, \frac{1}{c})\ , \\ n_{II} &= (-a,\, 0,\, b,\, \frac{-1}{c})\ . \end{aligned}$$

Condition (II) means that $A_I, \dots, C_{II}$ are constants. (I) is equivalent to $< n_I,\, P_I > = < n_{II},\, P_{II} > = 0$, i.e.

$$\begin{aligned} C_I &= -c\,(a\,A_I + b\,B_I)\ , \\ C_{II} &= c\,(a\,A_{II} + b\,B_{II})\ . \end{aligned} \tag{15}$$

If the magnitude of the force is denoted by K, one obtains from (III):

$$\begin{aligned} P_{II} - P_I &= (A_{II} - A_I,\, 0,\, 0,\, B_{II} - B_I,\, C_{II} - C_I) \\ &= K(-a,\, 0,\, 0,\, b,\, \frac{-1}{c}). \end{aligned} \tag{16}$$

From (15) and (16), one easily obtains

$$\begin{aligned} A_{II} &= A_I - 2a\,\frac{a\,A_I + b\,B_I}{a^2 - b^2 - \frac{1}{c^2}}\ , \\ B_{II} &= B_I + 2b\,\frac{a\,A_I + b\,B_I}{a^2 - b^2 - \frac{1}{c^2}}\ , \\ C_{II} &= (aA_I + b\,B_I)\,\frac{ca^2 - cb^2 + \frac{1}{c}}{b^2 + \frac{1}{c^2} - a^2}\ . \end{aligned}$$

So the mass m_{II} of the particle after having passed the edge can be compared to its original mass m_I:

$$\begin{aligned} m_{II}^2 &= A_{II}^2 - B_{II}^2 - C_{II}^2 = \\ &= A_I^2 - B_I^2 - C_I^2 + \frac{4C_I^2}{c^2(b^2 - a^2) + 1}\ . \end{aligned}$$

So the mass squared is increased by

$$\frac{4C_I^2}{c^2\,(b^2 - a^2) + 1}\ . \tag{17}$$

This effect is surprising. A pointlike particle really does change its mass in this simple model of a singularity. Notice that the denominator in (17) is zero iff n_I and n_{II} are light-like [see (15)].
The conditions (I)-(III) apply the laws of classical mechanics to pseudoeuclidean $\mathbb{R}^5$. Of course, there is no experimental justification for this besides the fact that it is compatible with General Relativity [4]:
In a flat embedding space, a particle is constrained to 4-dimensional space-time M. Therefore its 4-momentum must be tangent to M, and the "forces" responsible for its change turn out to be just the forces of constraint. In particular, they are normal to M.

Similar results could be obtained for the momentum and the mass of fields. One only applies the method of [4] to the present model of two hyperplanes.

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