Mean and scalar curvature homogeneous Riemannian manifolds

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Abstract

We introduce the notions of scalar curvature and mean curvature homogeneity for Riemannian manifolds and treat the problem whether such manifolds are locally homogeneous or not. Several partial results are derived.

1 Introduction

The general problem of determining *the extent to which the properties of certain families of geometric objects influence, or even characterize, the geometry (or curvature) of a (pseudo-)Riemannian manifold* (M, g) has evoked considerable interest over the years. (We refer to [5], [9], [13], [28], [29] for recent surveys and further references.) The geometric objects mentioned here are typically geodesic balls, spheres, tubes and disks. Their properties are studied when the manifold (M, g) is a so-called "model space", i.e., it has a high degree of symmetry. Two-point homogeneous spaces, locally symmetric spaces and particular or general locally homogeneous spaces can all serve as model spaces. On the other hand, one tries to determine which of these properties are characteristic for the model space under consideration. This approach has led to several geometric characterizations for the model spaces, but also to a number of interesting open problems, many of which turn out to be difficult to solve.

As an example, it is well-known that the volume of a ball of radius r in an n-dimensional Euclidean space is given by

$$V_m(r) = \omega r^n, \qquad \omega = \left((\frac{1}{2}n)!\right)^{-1} \pi^{(n/2)}. \tag{1}$$

In [17] A. Gray and the third author consider the converse problem: given a Riemannian manifold (M, g) such that the volume of its small geodesic balls is given by the formula (1), is (M, g) locally Euclidean ? Inspired by the lack of non-flat examples of Riemannian manifolds satisfying the volume property, they state the following so-called *volume conjecture*, which still remains open for the general case :

Conjecture 1.1 *Let (M^n, g) be a Riemannian manifold of dimension n and suppose that the volume of every geodesic ball of (sufficiently small) radius r is given by* (1). *Then (M, g) is locally Euclidean.*

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This problem is a particular case of the following more general one. Let (M, g) be a locally homogeneous Riemannian manifold. Then the volume of a geodesic ball $B_m(r)$ of (sufficiently small) radius r is independent of its center m. A Riemannian manifold satisfying this last property is said to be *ball-homogeneous* [22]. It was shown by P. Günther and F. Prüfer [19] that every *D'Atri space*, i.e., a Riemannian manifold all of whose local geodesic symmetries are volume-preserving up to sign (see, e.g., [25]), is ball-homogeneous. But all known examples of D'Atri spaces are locally homogeneous, and the same is true for ball-homogeneous spaces. We can therefore state the following generalization of the main open problem in the study of D'Atri spaces.

Question 1.2 *Let (M, g) be a ball-homogeneous Riemannian manifold. Is (M, g) locally homogeneous ?*

Both problems stated above have been investigated extensively but, although a lot of partial answers have been obtained (see, e.g., [6], [7], [8], [9], [12], [15], [25], [29]), to our knowledge, no definitive answer has been given yet.

In their study of the volume conjecture and related questions [10], B.Y. Chen and the third author remark that it is worthwhile to consider also functions other than the volume function which are naturally related to the geometry of geodesic spheres. The *total scalar curvature* and the *Euclidean total mean curvature* (as defined in [17]) are ready candidates. The first is an example of an intrinsic quantity, the second is extrinsic. (See Section 2.)

In [17], A. Gray and the third author study Riemannian manifolds all of whose (sufficiently small) geodesic spheres have the same Euclidean total mean curvature as those in a Euclidean space, and they prove that these manifolds are locally flat. Similarly, in [10], B.Y. Chen and the third author study Riemannian manifolds of dimension greater than three for which the total scalar curvature of the geodesic spheres has the same property, and they state a conjecture similar to the volume conjecture mentioned above.

A Riemannian manifold having the property that the total scalar curvature (respectively, the Euclidean total mean curvature) of the sufficiently small geodesic spheres does not depend on the center will be called *scalar curvature homogeneous* (respectively, *mean curvature homogeneous*), and the aim of this paper is to make a contribution to the study of these two classes of Riemannian manifolds. Clearly, locally homogeneous Riemannian manifolds satisfy both conditions.

Furthermore, it is easily seen that the (total) scalar curvature of every geodesic sphere in a two-dimensional Riemannian manifold vanishes identically and hence, every two-dimensional Riemannian manifold is scalar curvature homogeneous. It follows from the Gauss-Bonnet theorem (as we show in Theorem 3.1) that also any three-dimensional Riemannian manifold is scalar curvature homogeneous.

Apart from the low-dimensional cases mentioned above, no examples are known of scalar curvature homogeneous or mean curvature homogeneous Riemannian manifolds which are not locally homogeneous. Generalizing the central problem for ball-homogeneous spaces to the framework of scalar and mean curvature homogeneous spaces, we formulate the following two problems, on which we shall concentrate in the next sections.

Question 1.3 *Let (M, g) be a scalar curvature homogeneous Riemannian manifold with* $\dim M \geq 4$. *Is (M, g) locally homogeneous ?*

Question 1.4 *Let (M, g) be a mean curvature homogeneous Riemannian manifold. Is (M, g) locally homogeneous ?*

Finally, in view of the problems mentioned above, it would be interesting to investigate the following

Question 1.5 *Are the notions of ball-homogeneity, scalar curvature homogeneity (in dimension greater than three) and mean curvature homogeneity equivalent ?*

The rest of this paper is organized as follows. In Section 2, we introduce scalar curvature homogeneous and mean curvature homogeneous spaces and collect a number of results which will be used throughout the paper. In particular, we derive a number of necessary conditions for a manifold to be scalar curvature homogeneous or mean curvature homogeneous. In the subsequent sections, we study the questions stated above in a number of special cases. More precisely, we investigate the notions of scalar curvature and mean curvature homogeneity in the context of three-dimensional manifolds, conformally flat spaces, four-dimensional Einstein spaces, five-dimensional Sasakian manifolds and semi-symmetric spaces.

2 Preliminaries

Let (M^n, g) be an n-dimensional, smooth, connected Riemannian manifold. Denote by ∇ its Levi Civita connection and by R, ρ and τ the corresponding Riemann, Ricci and scalar curvature tensor, respectively. We use the following sign convention for R:

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all tangent vector fields X, Y on M.

We denote by $B_m(r)$ and $G_m(r)$ respectively the *geodesic ball* and *geodesic sphere* with center m and radius $r < i(m)$, where $i(m)$ denotes the injectivity radius at the point $m \in M$. More explicitly, this means

$$\begin{aligned} B_m(r) &= \{p \in M \mid d(m,p) \leq r\}, \\ G_m(r) &= \{p \in M \mid d(m,p) = r\}. \end{aligned}$$

Note that $G_m(r) = \exp_m(S^{n-1}(r))$, where $S^{n-1}(r) = \{x \in T_m M \mid \|x\| = r\}$ is the sphere of radius r in the tangent space to M at m.

Denote by $S_m(r)$ the $(n-1)$-dimensional volume of the geodesic sphere $G_m(r)$ and by $V_m(r)$ the volume of the geodesic ball $B_m(r)$. Then we have [17]

$$\begin{aligned} S_m(r) = \int_{G_m(r)} dp &= r^{n-1} \int_{S^{n-1}(1)} \theta_m(\exp_m(ru)) du \,, \\ V_m(r) &= \int_0^r S_m(t) dt \end{aligned}$$

where θ_m denotes the volume density function of the exponential map centered at m. The Riemannian manifold (M,g) is said to be *ball-homogeneous* [22] if $V_m(r)$, or equivalently $S_m(r)$, does not depend on the center $m \in M$ of the geodesic ball or sphere. See also [23].

Next, we consider the *Euclidean total mean curvature* $\mathcal{H}_m(r)$ and the *total scalar curvature* $\mathcal{T}_m(r)$ of small geodesic spheres $G_m(r)$. We have the following definitions [10], [17]:

$$\mathcal{H}_m(r) = r^{n-1}\int_{S^{n-1}(1)} h_m(\exp_m(ru))du\,, \tag{2}$$

$$\mathcal{T}_m(r) = \int_{G_m(r)} \tilde{\tau}_m(p)dp = r^{n-1}\int_{S^{n-1}(1)} (\tilde{\tau}_m\cdot\theta_m)(\exp_m(ru))du \tag{3}$$

where h_m denotes the mean curvature of $G_m(r)$ and $\tilde{\tau}_m$ its scalar curvature.

We define a Riemannian manifold (M,g) to be *mean curvature homogeneous* or *$\mathcal{H}$-homogeneous* (respectively, *scalar curvature homogeneous* or *$\mathcal{T}$-homogeneous*) if $\mathcal{H}_m(r)$ (respectively, $\mathcal{T}_m(r)$) does not depend on the point $m \in M$.

Note that $\mathcal{H}_m(r)\,r^{1-n}$ is the integral of the pull-back of the mean curvature of $G_m(r)$ to the unit sphere in the tangent space, which is not the same as the total mean curvature

$$\int_{G_m(r)} h_m(p)dp = r^{n-1}\int_{S^{n-1}(1)} (h_m\cdot\theta_m)(\exp_m(ru))du.$$

The last integral is just $\frac{d^2}{dr^2}V_m(r)$ [18]. For this reason and to avoid confusion, we called $\mathcal{H}_m(r)$ the *Euclidean* total mean curvature.

We shall make extensive use of the following power series expansions of $\mathcal{H}_m(r)$ and $\mathcal{T}_m(r)$ in terms of curvature invariants. For the sake of completeness, we also give the expansion of $V_m(r)$.

Proposition 2.1 *We have*

$$V_m(r) = \omega r^n\Big\{1 + V_2(m)\,r^2 + V_4(m)\,r^4 + V_6(m)\,r^6 + O(r^8)\Big\} \tag{4}$$

where

$$\begin{aligned}
V_2 &= -\frac{\tau}{6(n+2)},\\
V_4 &= \frac{1}{360(n+2)(n+4)}\Big\{-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\tau\Delta\tau\Big\},\\
V_6 &= \frac{1}{720(n+2)(n+4)(n+6)}\Big\{-\frac{5}{9}\tau^3 - \frac{8}{3}\tau\|\varrho\|^2 + \tau\|R\|^2 + \frac{64}{63}\check{\rho} - \frac{64}{21}\langle\rho\otimes\rho,\bar{R}\rangle\\
&\quad +\frac{32}{7}\langle\rho,\dot{R}\rangle - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} + \frac{45}{7}\|\nabla\tau\|^2 + \frac{45}{14}\|\nabla\rho\|^2 + \frac{45}{7}\alpha(\rho) - \frac{45}{14}\|\nabla R\|^2\\
&\quad +6\tau\Delta\tau + \frac{48}{7}\langle\Delta\rho,\rho\rangle + \frac{54}{7}\langle\nabla^2\tau,\rho\rangle - \frac{30}{7}\langle\Delta R,R\rangle - \frac{45}{7}\Delta^2\tau\Big\}.
\end{aligned}$$

For a proof and the definition of the curvature invariants involved, we refer to [17].

Proposition 2.2 *We have*

$$\mathcal{H}_m(r) = n\omega r^{n-2}\Big\{n - 1 + H_2(m)\, r^2 + H_4(m)\, r^4 + H_6(m)\, r^6 + O(r^8)\Big\} \tag{5}$$

where

$$\begin{aligned}
H_2 &= -\frac{\tau}{3n},\\
H_4 &= -\frac{1}{90n(n+2)}\Big\{3\|R\|^2 + 2\|\rho\|^2 + 18\Delta\tau\Big\},\\
H_6 &= \frac{1}{840n(n+2)(n+4)}\Big\{-\frac{412}{9}\check{\rho} + 44\langle\rho\otimes\rho, \bar{R}\rangle + 4\langle\rho, \dot{R}\rangle - \frac{110}{9}\check{R} - \frac{200}{9}\check{\bar{R}}\\
&\quad -25\alpha(\rho) - 45\Delta^2\tau - 25\|\nabla\tau\|^2 - 72\langle\nabla^2\tau, \rho\rangle - \frac{25}{2}\|\nabla\rho\|^2 + 6\langle\Delta\rho, \rho\rangle\\
&\quad -30\langle\Delta R, R\rangle - \frac{45}{2}\|\nabla R\|^2\Big\}.
\end{aligned}$$

This proposition corresponds to [17, Theorem 12.3].

Proposition 2.3 *We have*

$$\mathcal{T}_m(r) = n\omega r^{n-3}\Big\{(n-1)(n-2) + T_2(m)\, r^2 + T_4(m)\, r^4 + T_6(m)\, r^6 + O(r^8)\Big\} \tag{6}$$

where

$$\begin{aligned}
T_2 &= -\frac{(n-2)(n-3)}{6n}\tau,\\
T_4 &= \frac{1}{360n(n+2)}\Big\{-3(n+2)(n+3)\|R\|^2 + 8(n^2+5n+21)\|\rho\|^2 + 5(n^2-7n-6)\tau^2\\
&\quad -18(n-2)(n-3)\Delta\tau\Big\},\\
T_6 &= \frac{1}{n(n+2)(n+4)}\Big\{-\frac{n^2-9n-34}{1296}\tau^3 - \frac{n^2+3n+17}{270}\tau\|\rho\|^2\\
&\quad +\frac{1}{720}(n+1)(n+2)\tau\|R\|^2 + \frac{4n^2+36n-424}{2835}\check{\rho} - \frac{4n^2+36n-4}{945}\langle\rho\otimes\rho, \bar{R}\rangle\\
&\quad +\frac{2n^2+18n-2}{315}\langle\rho, \dot{R}\rangle - \frac{11n^2+99n-158}{4536}\check{R} - \frac{5n^2+45n-278}{1134}\check{\bar{R}}\\
&\quad +\frac{n^2-5n-22}{112}\|\nabla\tau\|^2 + \frac{n^2+9n+188}{224}\|\nabla\rho\|^2 + \frac{(n+12)(n-3)}{112}\alpha(\rho)\\
&\quad -\frac{(n+4)(n+5)}{224}\|\nabla R\|^2 + \frac{(n+2)(n-9)}{120}\tau\Delta\tau + \frac{n^2+9n+69}{105}\langle\Delta\rho, \rho\rangle\\
&\quad +\frac{(9n+38)(n-3)}{840}\langle\nabla^2\tau, \rho\rangle - \frac{n^2+9n+6}{168}\langle\Delta R, R\rangle - \frac{(n-2)(n-3)}{112}\Delta^2\tau\Big\}.
\end{aligned}$$

Proof. The coefficients T_2 and T_4 can be found in [10, Theorem 6.8]. Here, we indicate how to calculate the extra term T_6 in the expansion.

First, we have to compute the term of order 4 in the series expansion for the scalar curvature $\tilde{\tau}_m(p)$ of a small geodesic sphere $G_m(r)$ at a point $p = \exp_m(ru) = \gamma(r)$, where γ denotes the unit speed geodesic connecting m with p and $\gamma'(0) = u$. In [10, Theorem 4.4], this expansion is only known up to order 3.

For the components of the curvature tensor, the Gauss equation yields

$$\tilde{R}_{abcd}(p) = R_{abcd}(p) + \Big(\sigma_{ac}\sigma_{bd} - \sigma_{ad}\sigma_{bc}\Big)(p) \tag{7}$$

where σ denotes the second fundamental form of the geodesic sphere.

With respect to a parallel orthonormal frame field $\{E_a \,|\, a = 1, \ldots, n\}$ along γ such that $E_1 = \gamma'$, we have

$$R_{abcd}(p) = R_{abcd}(m) + r\,(\nabla_u R_{abcd})(m) + \ldots + \frac{r^4}{4!}\,(\nabla^4_{uuuu} R_{abcd})(m) + O(r^5).$$

So, expanding the first term on the right-hand side in (7) yields as fourth order coefficient $\frac{1}{24}\,(\nabla^4_{uuuu} R_{abcd})(m)$. For the second term, we use the known expansions for σ_{ab} in [10, Theorem 3.1]. Contracting (7) twice, we get the coefficient of r^4 in the expansion of $\tilde{\tau}_m(p)$.

Combining this with the expansion of θ_m in [10, Corollary 2.4], a routine computation gives the fourth order coefficient of $(\tilde{\tau}_m \cdot \theta_m)(p)$.

Now, in view of (3), this coefficient has to be integrated over the unit sphere $S^{n-1}(1)$ in the tangent space. To do this, one may use the method and fundamental formulas in the proof of [17, Theorem 3.3]. After a long computation, we get the result. ∎

For a manifold to be ball-homogeneous, $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous, the coefficients of their respective series expansions must be global constants. As an example, it follows immediately that the scalar curvature τ is always globally constant for spaces satisfying one of these homogeneity conditions (where $n > 3$ for $\mathcal{T}_m(r)$). This implies that any two-dimensional $\mathcal{H}$-homogeneous space has constant curvature and hence, is locally homogeneous.

Locally homogeneous spaces provide trivial examples of ball-homogeneous, $\mathcal{H}$-homogeneous and $\mathcal{T}$-homogeneous spaces. Moreover, Günther and Prüfer show in [19, Section 3] that every D'Atri space is also ball-homogeneous. Here we put the following

Question 2.4 *Is every D'Atri space $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous ?*

The answer is positive for the subclass of harmonic spaces but up to now we do not know an answer for the general case.

3 Three-dimensional manifolds

First we show that every three-dimensional manifold is $\mathcal{T}$-homogeneous. Hence, for the three-dimensional case, only ball- and $\mathcal{H}$-homogeneity are relevant.

Theorem 3.1 *Let (M^3, g) be a three-dimensional Riemannian manifold. Then the total scalar curvature of any sufficiently small geodesic sphere is equal to 8π.*

Proof. This follows by the Gauss-Bonnet theorem for two-dimensional compact Riemannian manifolds,

$$\mathcal{T}_m = \int_{G_m(r)} \tilde{\tau}_m dp = 4\pi\chi\Big(G_m(r)\Big),$$

since $\chi\Big(G_m(r)\Big) = 2$.

∎

Concerning Question 1.4 in the three-dimensional case, it seems to be difficult to obtain a complete answer. Nevertheless, it is possible to give some partial results.

First, we mention a result from flow geometry where the main topic is the study of Riemannian manifolds equipped with a Riemannian flow $\mathcal{F}_\xi$ generated by a unit Killing vector field ξ. See [14] ([16] gives a brief survey). Such a flow $\mathcal{F}_\xi$ is said to be *normal* if the curvature transformations R_{XY} satisfy $R(X,Y,X,\xi) = 0$ for all X, Y orthogonal to ξ. Note that these manifolds are generalizations of Sasakian manifolds [1], [30]. We have [15]

Theorem 3.2 *A three-dimensional Riemannian manifold equipped with a normal flow $\mathcal{F}_\xi$ is locally isometric to a naturally reductive space if and only if the scalar curvature is constant.*

Since $\mathcal{H}$-homogeneity implies constant scalar curvature, we get from this:

Corollary 3.3 *A three-dimensional Riemannian manifold equipped with a normal flow is locally isometric to a naturally reductive space if and only if it is $\mathcal{H}$-homogeneous.*

We can give more partial answers when the Ricci operator has at most two distinct eigenvalues or when the metric is Einstein-like. We first establish the following

Lemma 3.4 *Let (M^3, g) be a three-dimensional Riemannian manifold for which the curvature invariants τ, $\|\rho\|^2$, $a\,\check{\rho} + b\,\|\nabla\rho\|^2 + c\,\alpha(\rho)$ are globally constant on M (for some real constants a, b, c).*

If the Ricci tensor has at most two distinct eigenvalues and $b \neq 0$, then the manifold is locally homogeneous.

If the Ricci tensor is a Codazzi tensor and $a \neq 3(b+c)$, then the manifold is locally symmetric.

If the Ricci tensor is cyclic-parallel and $b \neq 0 \neq a - 3(c-2b)$, then the manifold is locally isometric to a naturally reductive homogeneous space.

For the proofs we refer to [7, Sections 3 and 4] where these results are obtained for ball-homogeneous spaces.

Lemma 3.5 *Let (M^3, g) be a three-dimensional Riemannian manifold. If (M^3, g) is ball-homogeneous or $\mathcal{H}$-homogeneous, then the curvature invariants τ and $\|\rho\|^2$ are constant on M. Moreover, the following invariants are also constant on M:*

$$256\check{\rho} + 9\|\nabla\rho\|^2 + 90\alpha(\rho) \tag{8}$$

for the ball-homogeneous case, and

$$8\check{\rho} + 207\|\nabla\rho\|^2 - 450\alpha(\rho) \tag{9}$$

for the $\mathcal{H}$-homogeneous case.

Proof. The constancy of τ follows immediately from the second term in the series expansion (4) and (5). The third term in these expansions, combined with the relation $\|R\|^2 = 4\|\rho\|^2 - \tau^2$, which holds for arbitrary three-dimensional manifolds, implies that $\|\rho\|^2$ and $\|R\|^2$ are constant on M. Finally, using the constancy of these three invariants together with the well-known relations between curvature invariants for three-dimensional spaces (see [17, (2.23)]) and the fact that

$$\frac{1}{2}\Delta\|R\|^2 = \langle\Delta R, R\rangle + \|\nabla R\|^2, \tag{10}$$

$$\frac{1}{2}\Delta\|\rho\|^2 = \langle\Delta\rho, R\rangle + \|\nabla\rho\|^2, \tag{11}$$

the fourth term in the expansions simplifies to the expressions (8) and (9) above.

∎

Now, the following theorems follow easily

Theorem 3.6 *A three-dimensional $\mathcal{H}$-homogeneous or ball-homogeneous space with at most two distinct Ricci eigenvalues is locally homogeneous.*

Theorem 3.7 *A three-dimensional manifold is locally symmetric if and only if it is $\mathcal{H}$-homogeneous or ball-homogeneous and its Ricci tensor is a Codazzi tensor.*

Corollary 3.8 *A three-dimensional manifold is locally symmetric if and only if it is $\mathcal{H}$-homogeneous or ball-homogeneous and conformally flat.*

Proof. This follows from the previous theorem and the fact that for τ constant, a three-dimensional manifold has a Codazzi Ricci tensor if and only if it is conformally flat.

∎

Theorem 3.9 *A three-dimensional manifold is locally isometric to a naturally reductive homogeneous space if and only if it is $\mathcal{H}$-homogeneous or ball-homogeneous and has cyclic-parallel Ricci tensor.*

4 Conformally flat manifolds

A theorem of Takagi [26] states that every locally homogeneous conformally flat space is locally symmetric. In view of Questions 1.3 and 1.4, it is worthwhile to see if this still holds when local homogeneity is replaced by $\mathcal{H}$-homogeneity or $\mathcal{T}$-homogeneity. This would also generalize part of Corollary 3.8. First, we have

Lemma 4.1 *Let (M^n, g), $n \geq 4$, be a conformally flat Riemannian manifold for which the curvature invariants τ, $\|\rho\|^2$, $a\,\check{\rho} + b\,\|\nabla\rho\|^2 + c\,\alpha(\rho)$ are globally constant on M (for some real constants a, b, c). If the Ricci tensor has at most three distinct eigenvalues and $a(n-2) - n(b+c) \neq 0$, then the manifold is locally symmetric.*

The proof proceeds as in [7, Theorem 5.1] where the ball-homogeneous case is considered. The constancy of the given curvature invariants, together with the limited number of Ricci eigenvalues, implies that these are constant and that the manifold is curvature homogeneous. The extension of Takagi's result to curvature homogeneous spaces (see [7, Proposition 4.1] and [21]) finishes the proof.

Now, Corollary 3.8 generalizes to

Theorem 4.2 *An n-dimensional ($n \geq 4$) conformally flat space with at most three distinct Ricci roots and which is also $\mathcal{H}$-homogeneous, $\mathcal{T}$-homogeneous or ball-homogeneous, is locally symmetric.*

Proof. Using the constancy of the fourth term in the series expansion (4) of $V_m(r)$, it is shown in the proof of [7, Theorem 5.1] that a particular expression of the form $a\,\check{\rho} + b\,\|\nabla\rho\|^2 + c\,\alpha(\rho)$ has to be constant on M, where the coefficients a, b, c satisfy the condition $a(n-2) - n(b+c) \neq 0$ in the previous lemma.

In the same way, we use the expansions (5) and (6), respectively for the $\mathcal{H}$-homogeneous and the $\mathcal{T}$-homogeneous case, to obtain similar expressions. For the $\mathcal{H}$-homogeneous case, the condition $a(n-2) - n(b+c) \neq 0$ amounts to the fact that the polynomial $41n^3 + 168n^2 - 436n - 2048$ does not have integer roots $n \geq 4$. For the $\mathcal{T}$-homogeneous case, the condition is realized for the polynomial $223n^4 + 660n^3 - 7100n^2 - 8400n + 116992$.

∎

5 Four-dimensional Einstein manifolds

A well-known theorem of Jensen [20] states that every four-dimensional locally homogeneous Einstein manifold is locally symmetric. So, if Questions 1.3 and 1.4 have a positive answer, then the result of Jensen should hold even for $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous Einstein manifolds. Here, we shall give positive answers when the manifold is in addition 2-stein, Kähler or Hermitian. The corresponding results for ball-homogeneous spaces are given in [6]. Recall that a Riemannian manifold (M^n, g) is said to be a 2-*stein space* if it is an Einstein space and if in addition $\sum_{a,b} R(X, e_a, X, e_b)^2 =$

$\lambda\, g(X,X)^2$ for any tangent vector X, where $\{e_a\}$ is an orthonormal basis of the tangent space.

First, we establish the following

Lemma 5.1 *Let (M^n, g) be an n-dimensional Einstein space which is $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous. For arbitrary $n \geq 4$, we have that $\|R\|^2$ is constant on M. For $n = 4$, also $\|\nabla R\|^2$ is constant on M.*

Proof. Note that for Einstein spaces with $n \geq 3$, the scalar curvature τ and the invariant $\|\rho\|^2$ are constant. Then, the constancy of $\|R\|^2$ follows immediately from the third term in (5) or (6).

Obviously, the curvature invariants $\|\nabla\tau\|^2$, $\|\nabla\rho\|^2$, $\alpha(\rho)$, $\tau\,\nabla\tau$, $\langle\Delta\rho,\rho\rangle$, $\langle\nabla^2\tau,\rho\rangle$ and $\Delta^2\tau$ vanish. Furthermore,

$$\begin{aligned} \check{\rho} = \langle\rho\otimes\rho, R\rangle &= \frac{\tau^3}{n^2}, \\ \langle\rho, \dot{R}\rangle &= \frac{\tau}{n}\|R\|^2, \end{aligned}$$

and hence, these invariants are clearly constant. For four-dimensional Einstein spaces, we have [17]

$$\begin{aligned} \check{R} &= -\frac{\tau^3}{12} + \frac{2\tau}{3}\|R\|^2 - \frac{1}{3}\langle\Delta R, R\rangle, \\ \check{\check{R}} &= \frac{\tau^3}{48} - \frac{\tau}{24}\|R\|^2 - \frac{1}{6}\langle\Delta R, R\rangle \end{aligned}$$

and, from (10), we get $\langle\Delta R, R\rangle = -\|\nabla R\|^2$.

Finally, combining these relations with the fourth term in the series expansions (5) and (6) yields the required result.

∎

As in the ball-homogeneous case [6], this lemma is the basic property needed to prove the following theorems. So, we refer to that paper for the proofs. An important technique is the use of Singer-Thorpe bases.

Theorem 5.2 *Let (M^4, g) be a four-dimensional 2-stein space which is $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous. Then (M, g) is flat or locally isometric to a rank-one symmetric space.*

Theorem 5.3 *Let (M^4, g) be a four-dimensional Kähler space (not necessarily Einstein). Then (M, g) is locally symmetric if and only if it is Ricci-parallel and $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous.*

Theorem 5.4 *Let (M^4, g) be a four-dimensional connected Hermitian Einstein space which is $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous. Then (M, g) is locally symmetric.*

6 Five-dimensional spaces

In this section, we give a Sasakian analog of Theorem 5.3. Detailed information on Sasakian geometry may be found in [1] and [30]. We remark that a Sasakian manifold is said to be *Ricci-η-parallel* if its Ricci tensor is parallel with respect to horizontal vectors, that is,

$$(\nabla_X \rho)(Y, Z) = 0$$

for all tangent vectors X, Y, Z satisfying $\eta(X) = \eta(Y) = \eta(Z) = 0$.

Theorem 6.1 *Let $(M^5, g, \varphi, \xi, \eta)$ be a five-dimensional Sasakian manifold. Then the manifold is locally φ-symmetric if and only if it is Ricci-η-parallel and $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous.*

Proof. A locally φ-symmetric space is a Sasakian manifold such that the local reflections with respect to the flow lines of the characteristic vector field are isometries. Such spaces are locally homogeneous and they are also Ricci-η-parallel (see [27]). So, they are clearly $\mathcal{H}$-homogeneous and $\mathcal{T}$-homogeneous.

For the converse, we proceed as in the proof of [12, Theorem 6]. Equation [12, (7)] must be replaced by a new one relevant to our cases. This is obtained by simplifying in a similar way the terms H_4 and T_4 in the expansions (5) and (6).

For the $\mathcal{H}$-homogeneous case, we obtain the same expression [12, (7)] as in the ball-homogeneous case, that is,

$$44\check{R} + 80\check{\check{R}} - 27\|\nabla R\|^2 = \text{const.}$$

For the $\mathcal{T}$-homogeneous case, we get

$$\frac{1}{4536}(11n^2+99n-158)\check{R}+\frac{1}{1134}(5n^2+45n-278)\check{\check{R}}-\frac{1}{672}(n-3)(n+12)\|\nabla R\|^2 = \text{const.}$$

Both expressions are of the form $a\,\check{R} + b\,\check{\check{R}} + c\,\|\nabla R\|^2 = \text{const.}$ Now, we combine each of these with [12, (8)] and [12, (9)], that is, respectively,

$$\begin{aligned} \check{R} + 4\check{\check{R}} - \|\nabla R\|^2 &= \text{const.}, \\ \check{R} - 2\check{\check{R}} &= \text{const.} \end{aligned}$$

The determinant of this system of three equations is equal to $2a + b + 6c$ and for both cases this is different from zero. We remark that for the $\mathcal{T}$-homogeneous case it is a multiple of $(n+4)(n+5)$. So, the invariants $\check{R}$, $\check{\check{R}}$ and $\|\nabla R\|^2$ are all constant. Using this, we then show that the Kählerian base spaces of the local fibrations are locally symmetric spaces and this characterizes local φ-symmetry. Again, we refer to the proof of [12, Theorem 6] for the details.

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Remark Five-dimensional ball-homogeneous spaces have also been treated in the flow geometric context in [15]. Similar theorems may be derived for $\mathcal{H}$- and $\mathcal{T}$-homogeneity, but we omit the details.

7 Semi-symmetric spaces

A Riemannian manifold (M^n, g) is said to be *semi-symmetric* if for each $m \in M$ the curvature tensor at m is the same as that of a symmetric space. This symmetric space may vary with the point m. See [4] for a thorough treatment and further references.

In view of Questions 1.3 and 1.4 and taking into account the fact that every semi-symmetric locally homogeneous space is locally symmetric [2, Proposition 3.1], it is natural to investigate whether this property still holds for semi-symmetric $\mathcal{H}$- or $\mathcal{T}$-homogeneous spaces. For the case of ball-homogeneity it has been proved to be true in [8]. Now, we give an affirmative answer for our two cases. In a similar way as for [8, Theorem 3.2 and Lemma 3.3] we first prove the following

Lemma 7.1 *Let (M^n, g) be a semi-symmetric space for which the curvature invariants τ and $a\|\nabla R\|^2 + b\|\nabla\rho\|^2 + c\,\alpha(\rho)$ are globally constant on M. Suppose also that $4a + b \neq 0$. Then the manifold is locally symmetric.*

Proof. Since the scalar curvature of the semi-symmetric manifold (M, g) is constant, (M, g) is locally isometric, on some open dense subset $\mathcal{U}$, to a Riemannian product of the form (see [24])

$$(M_S, g_S) \times (M_1, g_1) \times \cdots \times (M_r, g_r)$$

where the first factor is a symmetric space and the other ones are locally irreducible semi-symmetric Riemannian spaces foliated by totally geodesic leaves of codimension two with constant scalar curvature. The idea is to prove that also the factors $(M_1, g_1), \cdots, (M_r, g_r)$ are locally symmetric. For spaces of this type it is known (see [3], [4]) that there exists an open dense subset $\mathcal{U}_k$ of (M_k, g_k) such that around every point p of $\mathcal{U}_k$ we can choose local coordinates $(w, x^1, \ldots, x^{n-1})$ and an orthonormal coframe of the form

$$\begin{aligned} \omega^0 &= f(w, x^1)dw, \\ \omega^i &= dx^i + \sum_{j=1}^{n-1} D^i_j(w)x^j dw \quad (i = 1, \ldots, n-1) \end{aligned}$$

where $D^i_j(w) + D^j_i(w) = 0$ and $\tau = -2f^{-1}f''_{x^1x^1} \neq 0$. Using this local system, we obtain by a straightforward calculation

$$\begin{aligned} \|\nabla R\|^2 &= 2\tau^2 f^{-2}\sum_{j=1}^{n-1}(D^1_j)^2, \\ \|\nabla\rho\|^2 &= \frac{1}{2}\tau^2 f^{-2}\sum_{j=1}^{n-1}(D^1_j)^2, \\ \alpha(\rho) &= 0. \end{aligned} \tag{12}$$

Hence,

$$a\,\|\nabla R\|^2 + b\,\|\nabla\rho\|^2 + c\,\alpha(\rho) = (2a + \frac{1}{2}b)\tau^2 f^{-2}\sum_{j=1}^{n-1}(D_j^1)^2.$$

We stress that these computations hold for each of the semi-symmetric factors (M_k, g_k), but not for the manifold (M, g). Still, the curvature invariants τ and $a\,\|\nabla R\|^2 + b\,\|\nabla\rho\|^2 + c\,\alpha(\rho)$ are additive and this implies that for each factor (M_k, g_k) in the decomposition, the corresponding invariants are also constant on that factor.

Since $4a + b \neq 0$, it follows that $f^{-2}\sum_{j=1}^{n-1}(D_j^1)^2$ is constant. Since D_j^1 depends only on w and f depends on x^1, the constancy implies that $\sum_{j=1}^{n-1}(D_j^1)^2$ must be zero and therefore every term D_j^1 vanishes. It follows by (12) that $\|\nabla R\|^2 = 0$. This proves that every factor (M_k, g_k) is locally symmetric and as a consequence, also the product manifold (M, g).

∎

Lemma 7.2 *Let (M^n, g) be a semi-symmetric space. If (M, g) is $\mathcal{H}$-homogeneous or $\mathcal{T}$-homogeneous, then (M, g) is curvature homogeneous. Moreover, the curvature invariant*

$$15\|\nabla R\|^2 - 37\|\nabla\rho\|^2 - 50\alpha(\rho)$$

is constant for the $\mathcal{H}$-homogeneous case, and for the $\mathcal{T}$-homogeneous case,

$$(35n^2 + 315n + 420)\|\nabla R\|^2 - (47n^2 + 423n + 5028)\|\nabla\rho\|^2 - 30(n+12)(n-3)\alpha(\rho)$$

is constant.

Proof. Since τ is constant, (M, g) is curvature homogeneous on an open dense subset $\mathcal{U}$ (see [4] for details) and this implies that all scalar curvature invariants which do not involve derivatives of R are globally constant on M. This and (10), (11) simplify the fourth term in the series expansions of $\mathcal{H}_m(r)$ and $\mathcal{T}_m(r)$ and the results follow easily.

∎

The next theorem is obvious.

Theorem 7.3 *A semi-symmetric $\mathcal{H}$-homogeneous manifold (M^n, g) with $n \geq 3$ is locally symmetric. The same result holds for $\mathcal{T}$-homogeneous manifolds with $n \geq 4$.*

Proof. Invoke Lemma's 7.1 and 7.2 and check the condition $4a + b \neq 0$. (We note that for the $\mathcal{T}$-homogeneous case, $4a + b$ is proportional to $(n-3)(n+12)$.)

∎

It is interesting to look at the analogs of the volume conjecture stated in the Introduction, where we replace the volume function $V_m(r)$ of geodesic balls by the Euclidean mean curvature function $\mathcal{H}_m(r)$ or by the total scalar curvature function $\mathcal{T}_m(r)$ of small geodesic spheres.

First, we recall that the volume conjecture holds for locally symmetric spaces (see [11], [17]). In [8, Theorem 4.1], this result is extended to semi-symmetric spaces.

Next, the corresponding conjecture for the Euclidean total mean curvature holds for *arbitrary* Riemannian manifolds (see [17, Corollary 12.7]).

Using Theorem 7.3, we now prove that the corresponding conjecture for the total scalar curvature holds for semi-symmetric spaces. First, we prove the result for locally symmetric spaces.

Lemma 7.4 *Let (M^n, g), $n \geq 4$, be a locally symmetric space such that the total scalar curvature of small geodesic spheres satisfies the condition*

$$\mathcal{T}_m(r) = n(n-1)(n-2)\omega r^{n-3}\{1 + O(r^6)\}. \tag{13}$$

Then the manifold (M, g) is locally flat.

Proof. Condition (13), together with the series expansion (6) of $\mathcal{T}_m(r)$, implies

$$\begin{aligned} 0 &= \tau, \\ 0 &= -3(n+2)(n+3)\|R\|^2 + 8(n^2+5n+21)\|\rho\|^2. \end{aligned}$$

But the second equation can be rewritten as

$$(n+2)(n+3)\{-3\|R\|^2 + 8\|\rho\|^2\} + 120\|\rho\|^2 = 0.$$

From [11] and [17, Section 5] we know that for locally symmetric spaces we have the relation $-3\|R\|^2 + 8\|\rho\|^2 \geq 0$. So, we obtain that $\|\rho\|^2 = 0 = \|R\|^2$ and the manifold is locally flat.

∎

Theorem 7.5 *Let (M^n, g), $n \geq 4$, be a semi-symmetric space such that the total scalar curvature of small geodesic spheres is the same as in Euclidean space, that is,*

$$\mathcal{T}_m(r) = n(n-1)(n-2)\omega r^{n-3}. \tag{14}$$

Then the manifold (M, g) is locally flat.

Proof. A manifold satisfying this condition on $\mathcal{T}_m(r)$ is clearly $\mathcal{T}$-homogeneous. Hence, by Theorem 7.3, the manifold is locally symmetric. Applying the previous, we obtain the result.

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References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics 509, Springer-Verlag, Berlin, Heidelberg, NewYork, 1976

[2] E. Boeckx, *Einstein-like semi-symmetric spaces*, Arch. Math. (Brno) 29 (1993), 235–240

[3] E. Boeckx, O. Kowalski and L. Vanhecke, *Non-homogeneous relatives of symmetric spaces*, Diff. Geom. Appl. 4 (1994), 45–69

[4] E. Boeckx, O. Kowalski and L. Vanhecke, *Riemannian manifolds of conullity two*, World Scientific Publ. Co., Singapore, 1996

[5] P. Bueken and L. Vanhecke, *Volume and local homogeneity*, Proc. Workshop on Recent Topics in Differential Geometry, Santiago de Compostela 1997, (Eds. L.A. Cordero, E. García-Río), Public. Depto. Geometría y Topología, Univ. Santiago de Compostela, 89 (1998), 19–34

[6] G. Calvaruso, Ph. Tondeur and L. Vanhecke, *Four-dimensional ball-homogeneous and C-spaces*, Beiträge Algebra Geom. 38 (1997), 325–336

[7] G. Calvaruso and L. Vanhecke, *Special ball-homogeneous spaces*, Z. Anal. Anwendungen 16 (1997), 789–800

[8] G. Calvaruso and L. Vanhecke, *Semi-symmetric ball-homogeneous spaces and a volume conjecture*, Bull. Austral. Math. Soc. 57 (1998), 109–115

[9] G. Calvaruso and L. Vanhecke, *Ball-homogeneous spaces*, Proc. Workshop on Recent Topics in Differential Geometry, Santiago de Compostela 1997, (Eds. L.A. Cordero, E. García-Río), Public. Depto. Geometría y Topología, Univ. Santiago de Compostela, 89 (1998), 35–51

[10] B.Y. Chen and L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. 325 (1981), 28–67

[11] M. Ferrarotti and L. Vanhecke, *Locally symmetric spaces and the volume conjecture for geodesic balls*, Math. J. Toyama Univ. 14 (1991), 89–93

[12] E. García-Río and L. Vanhecke, *Five-dimensional φ-symmetric spaces*, Balkan J. Geom. Appl. 1 (1996), 31–44

[13] J. Gillard and L. Vanhecke, *Geodesic spheres, tubes and space forms*, Proc. Workshop on Recent Topics in Differential Geometry, Santiago de Compostela 1997, (Eds. L.A. Cordero, E. García-Río), Public. Depto. Geometría y Topología, Univ. Santiago de Compostela, 89 (1998), 165–176

[14] J. C. González-Dávila, M. C. González-Dávila and L. Vanhecke, *Reflections and isometric flows*, Kyungpook Math. J. 35 (1995), 113–144

[15] J. C. González-Dávila and L. Vanhecke, *D'Atri spaces and C-spaces in flow geometry*, Indian J. Pure Appl. Math. 29 (1998), 487–499

[16] J. C. González-Dávila and L. Vanhecke, *Local homogeneity in flow geometry*, Proc. Workshop on Recent Topics in Differential Geometry, Santiago de Compostela 1997, (Eds. L.A. Cordero, E. García-Río), Public. Depto. Geometría y Topología, Univ. Santiago de Compostela, 89 (1998), 177–192

[17] A. Gray and L. Vanhecke, *Riemannian geometry as determined by the volumes of small geodesic balls*, Acta Math. 142 (1979), 157–198

[18] N. Grossman, *On characterization of Riemannian manifolds by growth of tubular neighborhoods*, Proc. Amer. Math. Soc. 32 (1972), 556–560

[19] P. Günther and F. Prüfer, *Mean value operators, commuting differential operators and D'Atri spaces*, preprint, 1997

[20] G. R. Jensen, *Homogeneous Einstein spaces of dimension four*, J. Differential Geom. 3 (1969), 309–349

[21] O. Kowalski, *A note to a theorem by K. Sekigawa*, Comment. Math. Univ. Carolinæ 30 (1989), 85–88

[22] O. Kowalski and L. Vanhecke, *Ball-homogeneous and disk-homogeneous Riemannian manifolds*, Math. Z. 180 (1982), 429–444

[23] O. Kowalski and L. Vanhecke, *On disk-homogeneous symmetric spaces*, Ann. Global Anal. Geom. 1 (1983), 91–104

[24] O. Kowalski, F. Tricerri and L. Vanhecke, *Curvature homogeneous Riemannian manifolds*, J. Math. Pures Appl. 71 (1992), 471–501

[25] O. Kowalski, F. Prüfer and L. Vanhecke, *D'Atri spaces*, Topics in Geometry: In Memory of Joseph D'Atri (Ed. S. Gindikin), Progress in Nonlinear Differential Equations 20, 1996, Birkhäuser, Boston, Basel, Berlin, 241–284.

[26] H. Takagi, *Conformally flat Riemannian manifolds admitting a transitive group of isometries*, Tôhoku Math. J. 26 (1975), 103–110

[27] T. Takahashi, *Sasakian φ-symmetric spaces*, Tôhoku Math. J. 29 (1977), 91–113

[28] L. Vanhecke, *Geometry in normal and tubular neighborhoods*, Rend. Sem. Fac. Sci. Univ. Cagliari, Supplemento al Vol. 58 (1988), 73–176

[29] L. Vanhecke, *Scalar curvature invariants and local homogeneity*, Proc. Workshop on Differential Geometry and Topology (dedicated to the memory of Franco Tricerri), Palermo, 1996, Rend. Circ. Mat. Palermo Serie II, Suppl. 49 (1997), 275–287

[30] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics 3, World Scientific Publ. Co., Singapore, 1984

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