This paper is dedicated to memory of Ennio De Giorgi,
a Master both in science and in life.

p-Energy of a Curve on LIP–Manifolds and on General Metric Spaces[1]

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Abstract

On a general metric space (S, σ) three types of functionals $\mathcal{E}_h(\sigma, p)$ $(h = 1, 2, 3)$ [13], which generalize the concept of p–energy $(p \geq 1)$ of a curve, are considered. Conditions in order for them to coincide are studied. After giving a suitable notion of asymptotically equal generalized distances, one shows that if σ and ρ are so, then $\mathcal{E}_h(\sigma, p)(\gamma) = \mathcal{E}_h(\rho, p)(\gamma)$ for any $p \geq 1$, when the curve γ of S has finite energy for some $p_0 > 1$ [14]. A remarkable exemple is pointed out.

Introduction

Many problems of Calculus of Variations can be "transferred" from the framework of Euclidean spaces to the smooth manifolds one (e.g. length of curves, geodesics, Laplace-Beltrami operator,...).

Our intent was to see if it was possible to transfer some concepts to the Lipschitz manifolds (briefly *LIP manifolds*), i.e. topological manifolds in which changes of coordinates are given by Lipschitz mappings. Because a LIP mapping has (by Rademacher Theorem) partial derivatives that exist "almost everywhere" and which are measurable and bounded, the LIP manifolds are well suited for an investigation of the differential properties that might fail to be true on sets of zero measure, as emphasized by H. Whitney [42].

A LIP manifold, which generalizes a combinatorial manifold, may be envisaged as having, locally, the texture of a sand paper whose grains have curved, pyramidal non cuspidal faces.

We recall that, in dimension $\neq 4$, any topological manifold has a LIP structure, which is unique up to a LIP equivalence, arbitrary close to the identity [D. Sullivan, 1977]. There are, however, 4-dimensional topological manifolds with no, or more than one LIP structure [S. Donaldson, D. Sullivan, 1989].

D. Sullivan has shown that on any LIP manifold L_2-forms, exterior derivatives and currents may be defined. N. Teleman [37] has given a Hodge theory for *"Riemannian" LIP manifolds* (M, g) and has managed to construct the signature operator, used in order to extend to LIP category the index theory due to Atiyah-Singer in the smooth context ([38], [39]).

Actually, LIP manifolds and their generalization, *quasi conformal manifolds*, are studied unitarily in the framework of *non commutative geometry* [40].

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Metrics with singularities can be found in Physics, in Engineering: in the general case, the carrier is regular and the singularities of the metric have a physical meaning, e.g. they are caused by "irregularities" of the materials.

The first notion, which we have transferred, was the *"geodesic distance"* for a LIP Riemannian manifold (M, g), which concides with the usual one when M is smooth and g is a Riemannian metric.

The main difficulties are the following: the Jacobian of the chart transformations is defined outside an n-dimensional null set and the component of g, assumed only to be measurable, are defined up to an equivalence relation.

A first answer to the problem was given in a geometric way in 1988 ([8]). There we have constructed an intrinsic distance ϱ^g using the length $L(\gamma, g)$ of LIP curves γ that are *transversal* to a null set. Later (1991), starting from an idea of De Giorgi [16], we have defined a distance δ^g in an integral way, overcoming the above mentioned difficulties. Under very general hypotheses (M paracompact and oriented) the two distances ϱ^g and δ^g concide.

J.R. Norris [28] gives an alternative characterization of this distance function in terms of paths of least action, extending theorems on the small time asymptotics of the heat kernel to LIP Riemannian manifolds. Geodesic distance on singular Riemannian spaces, in particular on stratified spaces is considered recently by M. Ferrarotti [21].

Furthermore we prove that δ^g is locally equivalent to a Euclidean distance and (M, δ^g) is a *length space*. Then if (M, δ^g) is complete and locally compact, the Hopf–Rinow theorem in the general form ensures that two arbitrary points can be joined by a minimal geodesic ([22]), i. e. (M, δ^g) is a *geodesic space*. On these spaces it is possible to give a notion of curvature ([1]) and to define, in the case of Alexandrov spaces a sort of Riemannian structure ([29]).

Now let (M, δ) be a LIP (topological) manifold, where δ is a distance locally equivalent to a Euclidean one. We consider, φ, the *"directional derivative"* of δ at a point, which results a *"Finsler type"* metric. Then, if γ is an absolutely continuous curve, the usual length of γ in (M, δ) can be calculated by the integral of φ ([12]).

This led to the introduction [11] of the *Finslerian LIP manifolds* (M, F), which generalize, on one hand, the Finslerian (smooth) manifolds and, on the other hand, the Riemannian LIP manifolds.

Finslerian metrics in the classical case or in weaker hypotheses are studied by Busemann and Mayer and with generalization by Pauc [32] in view of the calculus of variations on metric spaces. In this context M. Amar and G. Bellettini ([3]) introduce a notion of total variation of a function depending on a possible discontinuous Finsler metric and study the connection with the theory of relaxation.

Finsler metrics are also connected with the symbols of elliptic operators having measurable coefficients ([27]).

Z. Shen ([34]) proves the one-to-one correspondence between C^0 length structure and C^0 Finsler structures on manifolds of class C^1. L. Tamássy [36] gives an explicit expression for k-dimensional ($1 \leq k \leq n$) area in a n-dimensional Finsler space and, with the aid of this notion of area, defines a curvature for submanifolds of a Finsler space endowed with a connection.

Now on LIP manifold (M, δ), where δ is a distance locally equivalent to a Euclidean one, it is possible to consider the n-dimensional Hausdorff measure and one proves a integral formula for the volume given by the Fisler structure induced by derivation of δ ([15]).

The above results suggest extensions and investigations in a very general framework, the *general metric spaces*, which are studied by Menger, Bouligand, Busemann, Pauc, Carathéodory and recently by Alexandrov and Gromov ([26], [6], [7], [32], [1], [22]).

A general metric space (S, σ) is a set S with a map

$$\sigma : S \times S \to [0, +\infty]$$

such that $\sigma(x, x) = 0$. The map σ expresses the intuitive idea of *"distance"*, i.e. the estimate of the gauge between two points.

This great generality is suggested by the following classical example: if (U, Φ) is a chart at the point $x \in M$, $\Phi(U) = V \subset R^n$ and $\xi = \Phi(x)$, then one can consider, for every $\eta \in V$, the function

$$\tilde{\sigma}(\xi, \eta) = \Big(\sum_{hk} g_{hk}(\xi)(\eta_h - \xi_h)(\eta_k - \xi_k)\Big)^{1/2}$$

which induces on M the function $\sigma(x, y) = \tilde{\sigma}(\Phi(x), \Phi(y))$, which generally is not a distance (because $\sigma(x, y) \neq \sigma(y, x)$ and the triangular property does not hold) but satisfies $\sigma(x, x) = 0$.

On general metric spaces we consider [13] three types of functionals $\mathcal{E}_h(\sigma, p)$ ($h = 1, 2, 3$), which generalize the concept of *p-energy* ($p \geq 1$) of a curve and we study conditions in order for them to coincide, hence there holds a theorem of integral representation of type

$$\mathcal{E}(\sigma, p)(\gamma; a, b) = \int_a^b \|\dot{\gamma}(t)\|^p dt$$

where $\mathcal{E}(\sigma, p)(\gamma)$ is the total p-variation of γ with respect to σ and $\|\dot{\gamma}(t)\|$ is the "norm of the derivative" of σ on $\gamma(t)$.

As in the classical case, minimizing the p-energy, one can introduce a "distance" $\delta_h(\sigma, p)$ (in general not symmetric), which, if $(S, \sigma) = (M, g)$ (or (M, F)), coincides whith the usual intrinsic distance induced by the Riemannian (or Finslerian) structure. If one repeats the procedure of construction of the functionals $\mathcal{E}_h$ by starting from the functions $\delta_h(\sigma, p)$, then

$$\delta_h(\delta_h(\sigma, p)) = \delta_h(\sigma, p)$$

i.e. stability for the distances holds.

These distances can be used in the study of evolution trajectories of figures with respect to an energy.

After giving a suitable notion of *asymptotically equal* generalized distances, we prove that if σ and ϱ are so, then

$$\mathcal{E}_h(\sigma, p)(\gamma) = \mathcal{E}_h(\varrho, p)(\gamma) \qquad \forall\, p \geq 1$$

when γ has finite energy for some $p_0 > 1$ ([14]).

The arguments on the generalized p-energy of a curve suggest a definition of energy for maps between metric spaces and we are able to introduce generalized Sobolev spaces ([25], [30], [23], [24]).

We conclude by considering a particular generalized distance $\sigma = \sigma_r$ $(r \geq 1)$ defined on the set S of the Lebesgue measurable parts of R

$$\sigma_r(A, B) = \Big(r \int_{A\Delta B} [\mathrm{dist}(x, \partial B)]^{r-1} dx\Big)^{1/r}, \quad r \geq 1$$

(where $A\Delta B$ is the symmetric difference between A and B).

The map σ_r, introduced by E. De Giorgi ([18]), as a generalization of σ_2, considered by Almgren-Taylor-Wang ([2]), is used in order to study the generalized minimizing motions (e.g. following the mean curvature) ([19], [31]).

Many of the present results are answers to some conjectures formulated in talks by E. De Giorgi, who in valuable discussions has drawn our attention to this problem of interactions among topology, differential geometry and calcul of variations.

1 Preliminaries

(1.1) A Lipschitz manifold M (briefly *LIP manifold*) of dimension n is a pair consisting of a topological manifold and an equivalence class of LIP atlases. A LIP atlas on M is a family of charts $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$, $(\alpha \in \Lambda)$ where $\{U_\alpha\}$ forms an open cover of M, $\Phi_\alpha : U_\alpha \to V_\alpha$ maps homeomorphically U_α onto a set V_α which is open either in R^n or in R^n_+ and for any α, β

$$\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \to \Phi_\beta(U_\alpha \cap U_\beta)$$

defines a Lipschitz homeomorphism.

(1.2) A Riemannian metric on M is a collection $g = \{g^\alpha\}$ where g^α is a Riemannian metric on $V_\alpha = \Phi_\alpha(U_\alpha) \subset R^n$, with measurable components, that satisfy the compatibility conditions

$$\Phi^*_{\alpha\beta}(g^\beta) = g^\alpha \qquad (\textit{no sum intended})$$

where the pull-back $\Phi^*_{\alpha\beta}$ is defined componentwise. The map $\Phi_{\alpha\beta}$ is LIP, so is differentiable a.e. and has bounded and measurable partial derivatives. Hence the hypotheses on g are admissible for any LIP manifold and in general one cannot ask for greater regularity.

(1.3) A Riemannian metric g will be called a *LIP Riemannian metric* on M if any g^α defines on V_α a L_2-norm which is equivalent to the standard L_2-norm. It is possible to find two strictly positive constants h_α and H_α, such that for every chart (U_α, Φ_α) and every n-ple $(v^1 ..., v^n) \in R^n$

$$h^2_\alpha \sum(v^i) \leq \sum g^\alpha_{ij}(z) v^i v^j \leq H^2_\alpha \sum(v^i) \qquad \textit{a.e. in } z.$$

i.e. g defines an "elliptic" metric.

(1.4) A *(weakly) Finslerian structure* on $V_\alpha \subset \boldsymbol{R}^n$ is a function

$$F_\alpha : V_\alpha \times \boldsymbol{R}^n \to \boldsymbol{R}$$

such that

1) $F_\alpha(\cdot, v)$ is measurable for any $v \in \boldsymbol{R}^n$ and $F_\alpha(\xi, \cdot)$ is continuous for a. e. $\xi \in V_\alpha$;
2) $F_\alpha(\xi, v) > 0$ for a. e. ξ if $v \neq 0$;
3) $F_\alpha(\xi, tv) = tF_\alpha(\xi, v)$ if $t > 0$.

(1.5) A *Finslerian structure* on M is a collection $F = \{F_\alpha\}$ of (weakly) Finslerian structures F_α on $V_\alpha = \Phi_\alpha(U_\alpha) \subseteq \boldsymbol{R}^n$, such that for any α, β the following compatibility condition

$$F_\alpha(\xi, v) = F_\beta(\Phi_{\alpha\beta}(\xi), (d\Phi_{\alpha\beta})(\xi)(v)) \qquad a.e.\ \xi \in V_\alpha,\ \forall v \in \boldsymbol{R}^n,$$

holds, where $\Phi_\alpha^{-1}(\xi) \in U_\alpha \cap U_\beta$.

(1.6) A Finsler structure F will be called a *LIP Finslerian metric* on M if for every α two (strictly) positive constants h_α and k_α exist such that

$$k_\alpha |v| \leq F_\alpha(\xi, v) \leq K_\alpha |v| \quad a.e.\ \xi \in V_\alpha$$

where $|v|$ is the standard Euclidean norm.

(1.7) A LIP manifold M with a LIP Finslerian metric F will be called simply a *LIP Finslerian manifold* (M, F). If $x = \Phi_\alpha^{-1}(\xi) \in U_\alpha$, sometimes one writes $F(x)$ instead of $F_\alpha(\xi, \cdot)$ and

$$\|v\|_{F(x)} = F_\alpha(\xi, v).$$

If (M, g) is a *LIP Riemannian manifold* and

$$F_\alpha(\xi, v) = (g^\alpha(\xi)(v, v))^{1/2} = \|v\|_{g(x)},$$

(M, g) is a LIP Finslerian manifold too.

2 Intrinsic distance

(2.1) Let (M, g) be a LIP Riemannian manifold and

$$Lip(x, y; M) = \{\gamma : [0, 1] \to M \quad LIP;\ \gamma(0) = x,\ \gamma(1) = y\}.$$

If $N \subset M$ is a null set (where the measure $|N|$ of N is defined through g), we set

$$\mathcal{N} = \{N \subset M;\ |N| = 0\}$$

and denote by

$$Lip_N(x, y; M) = \{\gamma \in Lip(x, y; M);\ \text{mis}\{t \in I;\ \gamma(t) \cap N\} = 0\}$$

the set of the *curves transversal* to N, which is not empty.

If $\gamma \in Lip_N(x,y;M)$ we introduce the integral

$$L_{\mathcal{A}}(\gamma,g) = \int_0^1 \sqrt{g_{ij}\dot\gamma^i\dot\gamma^j}\,dt = \int_0^1 \|\dot\gamma\|_{\gamma(t)}dt \tag{2.2}$$

with the understanding that $L_{\mathcal{A}}(\gamma;g) = +\infty$ whenever it does not exist. However with a suitable choice of the set $N = N(\mathcal{A})$ of zero measure it is possible to give a meaning to (2.2). Then we call $L_{\mathcal{A}}(\gamma;g)$ the *length of* γ (with respect to the atlas $\mathcal{A}$). Set

$$\varrho_N^g(x,y) = \inf\left\{L_{\mathcal{A}}(\gamma;g);\ \gamma \in Lip_N(x,y;M)\right\};$$

then

$$\varrho^g(x,y) = \sup\{\varrho_N(x,y);\ N \in \mathcal{N}\}$$

is a *distance* on M independent of $\mathcal{A}$ ([9]).

(2.3) Let x,y be two distinct points of M and

$$Lip(M;x,y) = \{u : M \to \boldsymbol{R}\ LIP;\ u(x)=0,\ u(y)=1\}.$$

If $x=y$ we put $\delta(x,y)=0$. If $x \neq y$ we put ([10])

$$\delta^g(x,y) = \limsup_{p\to\infty}\Big[\inf\Big\{\Big(\int_M \|du\|^p dv\Big)^{1/p};\ u \in Lip(M;x,y)\Big\}\Big]^{-1}$$

(which of course depends on the metric g).

One sees that δ^g is a distance which is LIP equivalent to the distance σ induced on M by the charts. Hence the completeness with respect to σ implies the completeness with respect to δ.

(2.4) **Theorem.** *Under our hypotheses on M,if (M,δ) is also complete, we have*

$$\delta(x,y)^{-1} = \min\{\|du\|_\infty;\ u \in Lip(M;x,y)\},$$

where $\|\cdot\|_\infty$ is the L_∞-norm.

Moreover if $u \in Lip(M) = \{v : M \to \boldsymbol{R};\ LIP\}$,

$$|u(x)-u(y)| \leq \|du\|_\infty \delta(x,y)$$

i.e. u is LIP also with respect to the metric δ of M.

Moreover the main result holds:

(2.5) **Theorem.** *If M is a paracompact oriented LIP manifold, there exists*

$$\lim_{p\to\infty}\left[\inf\{\|du\|_p;\ u \in Lip(M;x,y)\}\right] = \delta^g(x,y)^{-1}$$

and furthermore

$$\delta^g(x,y) = \left[\min\{\|du\|_\infty; u \in Lip(M;x,y)\}\right]^{-1}.$$

Moreover

$$\delta^g(x,y) = \varrho^g(x,y).$$

(2.6) **Theorem.** *If F^* is the Finslerian structure, dual of F, then*

$$\begin{aligned}\delta^F(x,y)^{-1} &= \inf\left\{\|F^*(df)\|_\infty;\ f \in Lip(M),\ f(x)=0,\ f(y)=1\right\},\\ \delta^F(x,y) &= \sup\left\{|f(x)-f(y)|;\ f \in Lip(M),\ \|F^*(df)\|_\infty \leq 1\right\}.\end{aligned}$$

Moreover there exists the following result (shown also by S. Venturini with different hypotheses [41]).

(2.7) **Theorem.** *Denote by $\mathcal{T}$ the set of the pseudodistances on M. If $\tau \in \mathcal{T}$ and y_0 is a fixed point of M, clearly $f(z) = \tau(z,y_0)$ is a LIP function. Then*

$$\delta^F(x,y) = \max\left\{\tau(x,y);\ \tau \in \mathcal{T};\ \|F^*(df)\|_\infty \leq 1,\ f(z) = \tau(z,y_0)\right\}.$$

3 From metric to Finslerian structure

(3.1) Let (M,δ) be a *LIP* (topological) manifold, where δ is a distance locally equivalent to a Euclidean one.

If (U,Φ) is a chart at the point $x \in M$, $\xi = \Phi(x)$, v is a vector of $V = \Phi(U) \subset R^n$, we consider the *"directional derivative"* of δ ([12]):

$$\varphi(\xi,v) = \lim_{t\to 0} \frac{\delta(\Phi^{-1}(\xi),\Phi^{-1}(\xi+tv))}{t}.$$

(3.2) **Theorem.** *The limit $\varphi(\xi,v)$ exists, the function $\varphi(.,v)$ is Borel-measurable, for a.e. ξ the function $\varphi(\xi,.)$ is a norm (dependent on ξ) and*

$$h|v| \leq \varphi(\xi,v) \leq k|v|,$$

hence φ defines a (general) Finsler structure on M.

Moreover if $\gamma : [a,b] \to M$ is an absolutely continuous curve, then for the *LIP* Finslerian manifold (M,φ)

$$\mathcal{L}(\gamma;\delta) = L(\gamma;\varphi),$$

holds, where

$$\mathcal{L}(\gamma;\delta) = \sup_T \left\{\sum_i \delta(\gamma(t_i),\gamma(t_{i+1}));\ T \text{ decomposition of } [a,b]\right\}$$

and

$$L(\gamma;\varphi) = \int_a^b \varphi(\gamma(t),\dot\gamma(t))dt.$$

Hence the length can be calculated by an integral.

(3.3) **Theorem.** *Under our hypotheses, let ϱ be the distance induced by φ. If one repeats the procedure starting from ϱ, then the derivative $\tilde{\varphi}$ of ϱ is a.e. equal to φ. Moreover if $\tilde{\varrho}$ is the distance induced by the Finsler structure $\tilde{\varphi}$, then*

$$\tilde{\varrho} = \varrho \ (= \varrho^{**} = \delta^{\varphi}),$$

where ϱ^{**} is the distance induced by φ^{**} and δ^{φ} the intrinsic one induced by φ.

(3.4) The metric structure, induced by δ, allows us to construct the n-dimensional Hausdorff measure, $\mathcal{H}^n_\delta$, while the Finslerian structure, induced by φ, allows us to define a notion of volume ([5], [36], where φ is more regular, at least continuous). Following a suggestion in [16], we give a theorem of integral representation, which binds the two concepts ([15]).

Let (M, δ) be a LIP manifold, where δ is a distance locally equivalent to a Euclidean one; if (U, Φ) is a local chart and $x, y \in U$, then $\xi = \Phi(x)$, $\eta = \Phi(y)$ are in $\Phi(U) = V \in R^n$ and the distance

$$d(\xi, \eta) = \delta(x, y)$$

is locally equivalent to the Euclidean one on R^n.

If $A \subset M$, we put by definition in any chart

$$\int_{A \cap U} dv = \int_{\Phi(A \cap U)} \frac{\omega_n}{|B(\xi, 1)|} d\xi$$

where ω_n is the volume of the unitary sphere with respect to the Euclidean metric and

$$|B(\xi, 1)| = meas\{v \in R^n; \ \varphi(\xi, v) \leq 1\}$$

where $meas$ means the Lebesgue measure and φ is the derivative of d.

(3.5) **Theorem.** *Under our hypotheses, for every $A \subset M$, s.t. $\Phi(A)$ is Lebesgue measurable, then*

$$\mathcal{H}^n_\delta(A) = \int_A dv.$$

(3.6) The previous result can be generalized. If $V \subset R^n$ is Lebesgue measurable, (X, δ) is a metric space and

$$f : V \to X$$

is a locally injective LIP map, the distance d on R^n defined by

$$d(\xi, h) = \delta(f(\xi), f(\eta))$$

is locally equivalent to the Euclidean and

$$\mathcal{H}^n_\delta(f(V)) = \int_V \frac{\omega_n}{|B(\xi, 1)|} d\xi$$

holds, where $B(\xi, 1)$ is built by

$$\varphi(\xi, v) = \lim_{t \to 0} \frac{\delta(f(\xi), f(\xi + tv))}{t}$$

(3.7) **Example.** Let S be the set of the intervals of $\boldsymbol{R}^n$ and

$$\Phi : R^2 \to S \qquad \Phi(a, b) =]\min\{a, b\},\ \max\{a, b\}[.$$

By putting

$$\mathcal{S} = \{(a, b) \in R^2;\ a \le b\},$$

the map $\Phi|\mathcal{S}$ is a one to one correspondence and S becomes a bidimensional manifold.

If $A \subset \mathcal{S}$ is a Lebesgue measurable set, we put for any $x, y \in A$, $r > 1$

$$d_r(x, y) = \min\left\{(|x_1 - y_1|^r + |x_2 - y_2|^r)^{1/r},\ (|x_2 - x_1| + |y_2 - y_1|)2^{(1-r)/r}\right\}.$$

Then one sees that

$$\mathcal{H}^2_{d_r}(A) = \mathcal{H}^2_{d_r}(\Phi(A)) = \pi|A|/C_r,$$

where

$$C_r = meas\,\{(v_1, v_2);\ |v_1|^r + |v_2|^r \le 1\} = 4\int_0^1 (1 - t^r)^{1/r} dt.$$

Other examples and generalizations are in [15].

4 p-Energy of a curve

(4.1) Let (S, σ) a *general metric space*, i.e. S is a set and

$$\sigma : S \times S \to [0, +\infty]$$

is a map s.t. $\sigma(x, x) = 0$. If $\gamma : [a, b] \to S$ is a (parametrized) *curve* of S, $a \le t' < t'' \le b$ and $T = \{t' = t_0 < t_1 < \dots < t_{n+1} = t''\}$ is a decomposition of $[t', t'']$, we define for $p \ge 1$, $p \in \boldsymbol{R}$, the following functionals, called *p-energy of the curve* γ,

$$\mathcal{E}_1(\sigma, p)(\gamma; t', t'') = \sup_T \Big\{\sum_{i=0}^n \frac{\sigma(\gamma(t_i), \gamma(t_{i+1}))^p}{(t_{i+1} - t_i)^{p-1}}\Big\};$$

$$\mathcal{E}_2(\sigma, p)(\gamma; t', t'') = \inf_T \Big\{\sum_{i=0}^n \mathcal{E}_1(\sigma, p)(\gamma; t_i, t_{i+1})\Big\};$$

$$\mathcal{E}_3(\sigma, p)(\gamma; t', t'') = {}^{*}\!\!\int_{t'}^{t''} \Big(\limsup_{h \to 0} \frac{\sigma(\gamma(t), \gamma(t + h))^p}{h^p}\Big) dt;$$

(where this latter integral is meant as a Lebesgue upper integral).

The functional $\mathcal{E}_1$ can be considered as the total p-variation of γ, with respect to the function σ. If σ is a *distance* and $p=1$ we have the usual concept of length of a curve, for $p=2$ we have the analogous concept of energy of curves even not smooth.

In the general case

$$\mathcal{E}_1 \geq \mathcal{E}_2 \geq \mathcal{E}_3$$

and there exist examples for which the inequalities hold strictly ([13]).

(4.2) We say that γ verifies the $(F\mathcal{E}_h)$ condition (briefly $\gamma \in (F\mathcal{E}_h)$), if some $p_0 > 1$ exists s.t. $\mathcal{E}_h(\sigma, p_0)(\gamma) < +\infty$.

(4.3) **Theorem.** *If σ verifies the triangular property (on $\gamma(I)$) then*

$$\mathcal{E}_1(\sigma,p)(\gamma) = \mathcal{E}_2(\sigma,p)(\gamma) \qquad \forall p \geq 1.$$

Moreover if $\gamma \in (F\mathcal{E}_1)$, then

$$\mathcal{E}_1(\sigma,p)(\gamma) = \mathcal{E}_2(\sigma,p)(\gamma) = \mathcal{E}_3(\sigma,p)(\gamma) = \mathcal{E}(\sigma,p)(\gamma) \qquad \forall p \geq 1.$$

(4.4) **Corollary.** *Let (S,σ) be an ordinary metric space and put*

$$\Gamma(x,y) = \{\gamma : [a,b] \to S;\ \gamma(a) = x,\ \gamma(b) = y, \gamma \in (F\mathcal{E})\}.$$

Then the functional

$$\mathcal{E}(\sigma,p) : \Gamma \to \boldsymbol{R}$$

is lower semicontinuous with respect to a.e. convergence, i.e. if (γ_h) is a sequence of curves s.t. $\gamma_h \to \gamma$ a.e. and $\mathcal{E}(\sigma,p)(\gamma_h) \leq c = const.$ for any h, then $\mathcal{E}(\sigma,p)(\gamma) \leq c$.

From the previous result it follows directly that $\mathcal{E}(\sigma,p)$ is lower semicontinuous even with respect to uniform convergence.

(4.5) **Corollary.** *$\mathcal{E}(\sigma,p)$ is lower semicontinuous with respect to L_p convergence induced on $\Gamma(x,y)$ by the distance*

$$d(\gamma_1,\gamma_2) = \Big(\int_a^b \sigma(\gamma_1(t),\gamma_2(t))^p dt\Big)^{1/p}.$$

(4.6) Classical examples are spaces (S,σ), where S is a smooth manifold (M,g) or (M,F) and σ is the intrinsic distance induced by g or F.

Now we consider that S is a Banach space and σ is the distance induced by the norm. Then

$$(F\mathcal{E}_1) \Rightarrow \gamma \text{ absolutely continuous} \Rightarrow \mathcal{E}(\sigma,p)(\gamma;a,b) = \int_a^b \|\dot{\gamma}(t)\|^p dt.$$

In non classical cases, the absolute continuity is replaced by the condition $(F\mathcal{E}_1)$, which permits to calculate the p-energy by integration.

5 The case of the LIP manifolds

(5.1) Let S a LIP (topological) manifold M and σ a distance δ locally equivalent to the Euclidean one. Then

$$\mathcal{E}(\delta,p)(\gamma;a,b)) = \int_a^b \varphi(\gamma,\dot\gamma)^p dt.$$

where φ is the *"derivative"* of δ.

In particular if S is a LIP Finslerian manifold of class C^1 and $\delta = \delta^F$ is the intrinsic distance induced by a *continuous norm* F then $\varphi = F$. If F is a generic Finslerian structure, then in general $\varphi \neq F$ holds, but $\varrho^\varphi = \varrho^F$.

(5.2) Let (M,F) be a LIP Finslerian manifold. If $\mathcal{A}$ is a LIP atlas of M, we remark that the integral

$$E_A(\gamma;F;p) = \int_a^b F(\gamma,\dot\gamma)^p dt$$

is not well defined for every curve γ and in general the value depends on the chosen atlas. If it does not exist or is not well defined we put $E_{\mathcal{A}}(\gamma;F;p) = +\infty$. The dependence on the atlas can be eliminated if we put by definition

$$E(\gamma;F;p) = \sup_N \Big\{ \liminf_{\tau\to\gamma} E_{\mathcal{A}}(\tau;F;p);\ \tau \in AC(M;x,y),\ \tau \pitchfork N,\ |N| = 0 \Big\}$$

where $AC(M;x,y)$ is the set of the absolutely continuous curves from $x = \gamma(a)$ to $y = \gamma(b)$ and $\tau \pitchfork N$ means that the curve τ is transversal to N, where $|N| = 0$. One can show that

(5.3) **Theorem.** *If δ^F is the intrinsic distance induced on M by F, then for every $p \geq 1$*

$$\mathcal{E}(\delta^F,p)(\gamma) = E(\gamma;F;p)$$

$$\inf\{E(\gamma;F;p);\ \gamma \in AC(M;x,y)\} = \delta^F(x,y)^p/(b-a)^{p-1}.$$

Therefore the geodesics are minimal curves of the functional $\mathcal{E}(\delta^F,p)$ for every $p \geq 1$. For $p = 1$, $\mathcal{E} = \mathcal{L}$ is the length of γ.

6 Stability of constructions

(6.1) Let (S,σ) be a general metric space. Put

$$\Gamma_1 = \{\gamma : [a,b] \to S;\ \gamma \in (F\mathcal{E}_1)\}$$

$$\Gamma_2 = \Gamma_3 = \{\gamma : [a,b] \to S;\ \gamma \in (F\mathcal{E}_2)\}$$

For $x, y \in S$ and $h = 1, 2, 3$, we define

$$\delta_h(\sigma, p)(x, y) = \inf_{\gamma}\{[\mathcal{E}_h(\sigma, p)(\gamma; a, b)(b-a)^{p-1}]^{1/p};\ \gamma(a) = x, \gamma(b) = y, \gamma \in \Gamma_h\}.$$

If $\delta_h(\sigma, p)$ is independent of p (as $h = 3$), one repeats the procedure of construction of the functionals $\mathcal{E}_h(\sigma, p)$ by starting from the functions $\delta_h(\sigma, p)$, then

$$\mathcal{E}_h(\delta_h(\sigma, p), p) \leq \mathcal{E}_h(\sigma, p),\ \ \delta_h(\delta_h(\sigma, p), p) = \delta_h(\sigma, p),$$

whence, by setting $\varrho_h(\sigma, p) = \delta_h(\delta_h(\sigma, p), p)$,

$$\mathcal{E}_h(\varrho_h(\sigma, p), p) = \mathcal{E}_h(\delta_h(\sigma, p), p),$$

i.e. the construction is stable.

(6.2) **Example.** Let $(S, \|\cdot\|)$ be a Banach space and

$$\sigma(x, y) = \|x - y\|^\alpha \qquad (\alpha > 0),$$

which is a distance for $\alpha \leq 1$.

If $\gamma(t) = x + t(y - x)$ is a fixed curve, then for any $p \geq 1$:

$$\begin{array}{ll} \mathcal{E}_2 = \mathcal{E}_3 = 0 & \alpha > 1, \\ \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \|x-y\|^p & \alpha = 1, \\ \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = +\infty & \alpha < 1, \end{array}$$

whence

$$\begin{array}{ll} \delta_2(x, y) = \delta_3(x, y) = 0 & \alpha > 1, \\ \delta_h(x, y) = \|x - y\| & \alpha = 1, \\ \delta_h(x, y) = \begin{cases} +\infty & x = y \\ 0 & x \neq y \end{cases} & \alpha < 1. \end{array}$$

7 Asymptotically equal generalized distances

(7.1) The notion of generalized distance, although it is poor, permits defining in the spaces (S, σ) the concept of "limit of a sequence".

An element $x \in S$ is called a *limit* of a sequence $\{x_k\}$ of elements of S (briefly $x_k \xrightarrow[\sigma]{} x$) if and only if

$$\min\Big\{\lim_k \sigma(x_k, x),\ \lim_k \sigma(x, x_k)\Big\} = 0.$$

Observe that an infinite sequence of elements of (S, σ) might have a limit without being a Cauchy sequence ([4]).

(7.2) Now let σ and ϱ be two *generalized* distances. We say ([14]), that σ is *asymptotically equal* to ϱ at $x \in S$ if, and only if,

$$x_k \xrightarrow[\sigma]{} x,\ y_k \xrightarrow[\sigma]{} x \Rightarrow \lim_k \sigma(x_k, y_k)/\varrho(x_k, y_k) = 1.$$

If σ is asymptotically equal to ϱ at any point $x \in S$, then we write $\sigma \sim \varrho$. In general $\sigma \sim \varrho$ does not imply $\varrho \sim \sigma$, as it is shown by the following example

$$\varrho(x, y) = |\sin \sigma(x, y)|,$$

where σ might be a distance in the usual sense.

If $\sigma \sim \varrho$ and a real number $a > 0$ exists such that

$$a\sigma(x, y) \leq \varrho(x, y)$$

then $\varrho \sim \sigma$.

(7.3) **Example** Let (M, δ) be a *LIP* manifold, where δ is a distance locally equivalent to a Euclidean one. By Theorem (3.2), for a.e. ξ the norm $\varphi(\xi, \cdot)$ induces on $\boldsymbol{R}^n$ the generalized distance

$$\tilde{d}(\xi, \eta) = \varphi(\xi, \eta - \xi).$$

One proves that a.e. δ is asymptotically equal to d, where

$$d(x, y) = \tilde{d}(\xi, \eta), \qquad x = \Phi^{-1}(\xi), \quad y = \Phi^{-1}(\eta).$$

(7.4) **Theorem.** *Let σ and ϱ be generalized distances and $\sigma \sim \varrho$. If γ is a curve of S s.t. $\mathcal{E}_3(\sigma, 1)(\gamma) < +\infty$, then $\mathcal{E}_3(\varrho, 1)(\gamma) < +\infty$ and*

$$\mathcal{E}_3(\sigma, p)(\gamma) = \mathcal{E}_3(\varrho, p)(\gamma) \qquad \forall p \geq 1.$$

If $\mathcal{E}_2(\sigma, p_0)(\gamma) < +\infty$ for some $p_0 > 1$, then $\mathcal{E}_2(\sigma, p_0)(\gamma) < +\infty$ too and

$$\mathcal{E}_2(\sigma, p)(\gamma) = \mathcal{E}_2(\varrho, p)(\gamma) \qquad \forall p \geq 1.$$

(7.5) **Theorem.** *Let S be a topological space and σ a continuous map. If $\sigma \sim \varrho$, then for every continuous curve*

$$\mathcal{E}_h(\sigma, p)(\gamma) = \mathcal{E}_h(\varrho, p)(\gamma) \qquad \forall p \geq 1,\ h = 2, 3.$$

Remarks.

(7.6) A statement analogous to (7.4) is not true for $\mathcal{E}_1$. Namely, if $S = \boldsymbol{R}$, $\gamma(t) = t$ with $a \leq t \leq b$

$$\sigma(x, y) = |x - y|, \qquad \varrho(x, y) = e^{\sigma(x,y)}\sigma(x, y)$$

then $\sigma \sim \varrho$, $\varrho \sim \sigma$, but

$$\mathcal{E}_1(\varrho, p)(\gamma) = e^{p(b-a)}(b - a) > (b - a) = \mathcal{E}_1(\sigma, p)(\gamma).$$

(7.7) The condition $\mathcal{E}_2(\sigma,1)(\gamma) < +\infty$ does not imply the equality $\mathcal{E}_2(\sigma,p)(\gamma) = \mathcal{E}_2(\varrho,p)(\gamma)$ even for finite energies. For example, if

$$\gamma(t) = \begin{cases} \bar{x} & a \le t \le c \\ \bar{y} & c \le t \le b \end{cases}$$

and $\sigma(\bar{x},\bar{y}) \neq \varrho(\bar{x},\bar{y})$, we have

$$\mathcal{E}_2(\sigma,1)(\gamma) = \sigma(\bar{x},\bar{y}) \neq \varrho(\bar{x},\bar{y}) = \mathcal{E}_2(\varrho,1)(\gamma).$$

8 A remarkable example

(8.1) Let $\tilde{S}$ be the set of the Lebesgue measurable parts of R^n and for every $A, B \in \tilde{S}$, define

$$\sigma_r(A,B) = \left(r \int_{A\Delta B} [\operatorname{dist}(x,\partial B)]^{r-1} dx\right)^{1/r} \qquad (r \ge 1)$$

(where $A\Delta B$ is the symmetric difference between A and B).

Clearly $(\tilde{S},\sigma_r)$ is a general metric space. When we identify two sets A and B s.t. $|A\Delta B| = 0$, σ_1 is the *Nikodým distance*, while σ_r $(r > 1)$ is not a distance.

Since for any $A, B \in \tilde{S}$ one may construct a curve γ joining A with B s.t.

$$\mathcal{E}_2(\sigma,p)(\gamma) = \mathcal{E}_3(\sigma,p)(\gamma) = 0 \Rightarrow \delta_2(A,B) = 0 \qquad \forall A, B \in \tilde{S},$$

it is convenient to restrict $\tilde{S}$ to a more meaningful subset

$$S = \{X \subset R^n;\ X \ \text{ convex and bounded.}\}$$

Now for any $A, B \in S$, with the above identification,

$$\sigma_r(A,B) = 0 \Rightarrow A = B.$$

Moreover (S,σ_1) is a topological space and we are able to prove that σ_r is continuous with respect to the two arguments ([30]):

(8.2) **Theorem.** *Let $A, B \in S$ with $|A| \neq 0$, $|B| \neq 0$. If $\{A_k\}$, $\{B_k\}$ are sequences in S and $\sigma_1(A_k,A) \to 0$, $\sigma_1(B_k,B) \to 0$, then*

$$\sigma_r(A_k,B_k) \to \sigma_r(A,B).$$

(8.3) **Theorem.** *Let $\{A_k\}$ be a sequence of S and $A \in S$. If $\sigma_r(A_k,A) \to 0$, then $\sigma_1(A_k,A) \to 0$. If $\sigma_r(A,A_k) \to 0$, then $\sigma_1(A,A_k) \to 0$.*

From the previous theorems it follows that, if $\gamma : [a,b] \to S$ is a curve joining A and B s.t. $\mathcal{E}_2(\sigma_r,p_0)(\gamma) < +\infty$ and $meas\{\gamma(t);\ t \in (\alpha,\beta)\} \neq 0$, then in the interval (α,β)

the curve γ is continuous with respect to the metric σ_1. Now γ can be reparametrized by the arclength and

$$\delta_2(\sigma_r,p)(A,B)=\inf\{[\mathcal{E}_2(\sigma_r,p)(\gamma;a,b)(b-a)^{p-1}]^{1/p};\ \gamma(a)=A,\gamma(b)=B\}$$

is independent of p.

(8.4) **Remark.** Any curve $\gamma:[0,1]\to S$ can be considered as an *evolution* from the figure $A=\gamma(0)$ to the figure $B=\gamma(1)$. Geodesics for distances $\delta_h(\sigma_r,p)$ are meaningful evolutions.

For example if S is the set of the intervals of R and $I_i,I_j\in S$, then one proves that ([13])

$$\delta_2(\sigma_r)(I_i,I_j)=\delta_3(\sigma_r)(I_i,I_j)=\min\{\ell_1,\ell_2\},$$

where, by putting $I_i=[a_i,b_i]$,

$$\ell_1=(|a_j-a_i|^r+|b_j-b_i|^r)^{1/r},$$

$$\ell_2=(|b_i-a_i|+|b_j-a_j|)2^{(1-r)/r}.$$

These quantities can be interpreted as the length of the curve γ_1 (resp. γ_2), where

$$\gamma_1(t)=[a_i+t(a_j-a_i),b_i+t(b_j-b_i)]$$

$$\gamma_2(t)=\begin{cases}[a_i+t(b_i-a_i),b_i+t(a_i-b_i)] & t\in[0,1/2]\\ [b_j+t(a_j-b_j),a_j+t(b_j-a_j)] & t\in]1/2,1]\end{cases}$$

realize the simplest evolutions of the interval I_i on I_j.

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