What is the Role of Twistors in Supergeometry?

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Abstract

The present is an extended version of the talk given by the author on the occasion of the "3rd International Workshop on Differential Geometry and its Applications and 1st German-Romanian Seminar on Geometry" (Sibiu 1997). It is based on joint work with D.V. Alekseevsky, C. Devchand and U. Semmelmann, see [1] and [2].

A supermanifold M is canonically associated to any pseudo Riemannian spin manifold (M_0, g_0). Its structure sheaf is the sheaf of local sections of the exterior algebra $\wedge S$ over the spinor bundle $S \to M_0$. M carries a G-structure $\mathcal{F}_G$ for a certain linear supergroup G whose even part is the spinor group. G is constructed using the coadjoint representation of a super extended Poincaré group. For any spinor field s on M_0 there is a corresponding odd vector field X_s on M. The twistor equation has the following supergeometric interpretation: X_s is an infinitesimal automorphism of the G-structure $\mathcal{F}_G$ iff S is a solution of the twistor equation.

1 Spin geometry

Let (M, g) be an oriented pseudo Riemannian manifold of signature (k, l) and denote by $P_{SO} \to M$ the $SO(k, l)$-principal bundle of positively oriented orthonormal frames. (M, g) is called a *spin manifold* if there is a $Spin(k, l)$-equivariant lift of P_{SO} to a $Spin(k, l)$-principal bundle $P_{Spin} \to M$. The *spinor bundle* is the vector bundle $S = P_{Spin} \times_\rho \Sigma$ over M associated to the spinor representation $\rho : Spin(k, l) \to GL(\Sigma)$. The Levi-Civita connection on M induces a connection on P_{Spin} and hence on all associated vector bundles; in particular, S carries a covariant derivative ∇. Finally, we have the Clifford multiplication $\mu : TM \otimes S \to S$ or equivalently the embedding $\iota_\mu : S \to T^*M \otimes S$, which turns each fibre S_p into a module over the Clifford algebra $C\ell(T_pM)$, $p \in M$. We have the canonical decomposition

$$T^*M \otimes S = ker\, \mu \oplus \iota_\mu(S)\,;$$

here we have identified TM and T^*M via g. Correspondingly there are two fundamental differential operators: the *Dirac operator*

$$D : \Gamma(S) \to \Gamma(S)$$

and the *twistor operator*

$$\mathcal{D} : \Gamma(S) \to \Gamma(\ker \mu)$$

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defined by the equation

$$\nabla = \mathcal{D} - \frac{1}{m}\iota_\mu \circ D\,,$$

where $m = k + l = \dim M$. A spinor field $s \in \Gamma(S)$ is called a *twistor* if it is a solution of the twistor equation $\mathcal{D}s = 0$. From the definition of the twistor operator $\mathcal{D}$ it follows that s is a twistor iff there exists a spinor field $\tilde{s}$ such that $\nabla s = \iota_\mu(\tilde{s})$, i.e. if

$$\nabla_X s = X \cdot \tilde{s} := \mu(X \otimes \tilde{s})$$

for all vector fields X on M.

2 Supergeometry

We consider now pairs $M = (M_0, \mathcal{A})$, where M_0 is a C^∞-manifold and $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras on M_0; $\dim M_0 = m$. M is called a *supermanifold* of dimension $m|n$ if it is locally isomorphic to the following model: Let $E \to M_0$ be a vector bundle of rank n and $\mathcal{E}$ the corresponding sheaf of local sections. Then the exterior sheaf $\wedge\mathcal{E} = \wedge^{ev}\mathcal{E} + \wedge^{odd}\mathcal{E}$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras on M_0 and $M(E) := (M_0, \wedge\mathcal{E})$ is the *standard supermanifold* associated to the vector bundle E. It is known that any supermanifold is globally (although not canonically) isomorphic to a standard supermanifold.

Let $\mathbf{A}$ be a $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra. A *derivation* of degree $\alpha \in \mathbb{Z}_2 = \{0, 1\}$ is an $\mathbb{R}$-linear map $X : \mathbf{A} \to \mathbf{A}$ satisfying the following Leibniz rule:

$$X(fg) = X(f)g + (-1)^{\alpha\tilde{f}} f X(g)$$

for all homogeneous $f, g \in \mathbf{A}$, where $\tilde{f}$ is the $\mathbb{Z}_2$-degree of f. Of course, derivations of degree 0 (respectively 1) are called even (respectively odd). A derivation of $\mathbf{A}$ is by definition a sum of an even and an odd derivation. The $\mathbb{Z}_2$-graded vector space of all derivations of $\mathbf{A}$ is denoted by $Der\,\mathbf{A} = (Der\,\mathbf{A})_0 + (Der\,\mathbf{A})_1$.

Let $M = (M_0, \mathcal{A})$ be a supermanifold of dimension $m|n$. Te correspondence $U \mapsto Der\,\mathcal{A}(U)$, $U \subset M_0$ open, defines a locally free sheaf of ($\mathbb{Z}_2$-graded) $\mathcal{A}$-modules on M_0.

Definition 1 *The sheaf $\mathcal{T}_M := Der\,\mathcal{A}$ is called* **tangent sheaf** *of M. (Local) sections of $\mathcal{T}_M$ are called (local)* **vector fields** *on M. A* **frame field** *over $U \subset M_0$ is a basis $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ of the $\mathcal{A}(U)$-module $\mathcal{T}_M(U)$; the X_i are even and the Y_j are odd local vector fields.*

3 G-Structures on supermanifolds

Recall that a $\mathbb{Z}_2$-graded algebra $\mathbf{A}$ is said to be *supercommutative* if the relation $ab = (-1)^{\tilde{a}\tilde{b}}ba$ holds for all homogeneous elements $a, b \in \mathbf{A}$ of degree $\tilde{a}, \tilde{b}$. In the following $\mathbf{A}$ will always denote a $\mathbb{Z}_2$-graded associative and supercommutative $\mathbb{R}$-algebra;

short a *superalgebra*. The $\mathbb{Z}_2$-graded vector space of rank $m|n$ is denoted by $\mathbb{R}^{m|n}$ and, more generally, the free $\mathbf{A}$-module of rank $m|n$ by $\mathbf{A}^{m|n} = \mathbb{R}^{m|n} \otimes \mathbf{A}$. Finally, $GL_{m|n}(\mathbf{A})$ stands for the group of even invertible elements of $End_{\mathbf{A}}(\mathbf{A}^{m|n})$.

Definition 2 *A* **supergroup** G *is a contravariant functor* $M \mapsto G(M)$ *from the category of supermanifolds into that of groups.*

Example: The *general linear supergroup* $M \mapsto GL_{m|n}(M) = GL_{m|n}(\mathcal{A}(M_0))$ is the composition of two functors:

$$\text{supermanifolds} \quad \overset{\text{(contravariant)}}{\longrightarrow} \quad \text{superalgebras} \quad \overset{\text{(covariant)}}{\longrightarrow} \quad \text{groups}$$

$$M = (M_0, \mathcal{A}) \quad \mapsto \quad \mathbf{A} = \mathcal{A}(M_0) \quad \mapsto \quad GL_{m|n}(\mathbf{A})\,.$$

A supergroup G is called *linear* if $G \subset GL_{m|n}$.

Definition 3 *A* **super Lie algebra** *is a* $\mathbb{Z}_2$*-graded vector space* $\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1$ *together with a bilinear map* $[\cdot,\cdot] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ *(super Lie bracket) such that:*

i) $[\mathbf{g}_\alpha, \mathbf{g}_\beta] \subset \mathbf{g}_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2$,

ii) $[x,y] = -(-1)^{\tilde{x}\tilde{y}}[y,x]$ *(skew symmetry) and*

iii) $[x,[y,z]] = [[x,y],z] + (-1)^{\tilde{x}\tilde{y}}[y,[x,z]]$ *(Jacobi identity) for all homogeneous elements* $x, y, z \in \mathbf{g}$.

Example: The supercommutator $[a,b] := ab - (-1)^{\tilde{a}\tilde{b}}ba$ defines on the (naturally) $\mathbb{Z}_2$-graded algebra $End\,\mathbb{R}^{m|n}$ the structure of super Lie algebra, which is denoted by $\mathbf{gl}_{m|n}$. Subalgebras $\mathbf{g} \subset \mathbf{gl}_{m|n}$ are called *linear super Lie algebras*.

For any linear super Lie algebra $\mathbf{g} \subset \mathbf{gl_{m|n}}$ there is a linear supergroup denoted by $\exp \mathbf{g}$. In fact, if $M = (M_0, \mathcal{A})$ is a supermanifold we define the group

$$(\exp \mathbf{g})(M) := \langle (\mathbf{g} \otimes \mathcal{A}(M_0))_0 \rangle \subset GL_{m|n}(\mathcal{A}(M_0))\,.$$

Note that the even part $(\mathbf{g} \otimes \mathcal{A}(M_0))_0$ of the super Lie algebra $\mathbf{g} \otimes \mathcal{A}(M_0)$ is a matrix Lie algebra for which the usual exponential series is defined and converges in the standard Fréchet topology of $\mathcal{A}(M_0) \cong \Gamma(\wedge E)$ (here E is a vector bundle such that $M \cong M(E)$).

Let now $M = (M_0, \mathcal{A})$ be a supermanifold, $\dim M = m|n$, $\mathcal{F}$ the sheaf of local frame fields and $G \subset GL_{m|n}$ a linear supergroup. The group $GL_{m|n}(\mathcal{A}(U))$ acts naturally from the right on the set $\mathcal{F}(U)$, $U \subset M_0$ open.

Definition 4 *A subsheaf* $\mathcal{F}_G \subset \mathcal{F}$ *is called* **G-structure** *if all* $p \in M_0$ *have an open neighborhood* U *such that* $G(M|_U) \subset GL_{m|n}(\mathcal{A}(U))$ *acts simply transitively on* $\mathcal{F}_G(U)$. *Here* $M|_U$ *stands for the supermanifold* $(U, \mathcal{A}|_U)$.

Example: $\mathcal{F}$ is always a $GL_{m|n}$-structure.

We have a natural left-action of vector fields X on $(m+n)$-tuples of vector fields $F = (X_1, \ldots X_{m+n})$:

$$F \mapsto L_X F := ([X, X_1], \ldots [X, X_{m+n}])\,,$$

where $[X,Y] := XY - (-1)^{\tilde{X}\tilde{Y}}YX$ is the supercommutator of vector fields.

Definition 5 *X is called an* **infinitesimal automorphism** *of the G-structure $\mathcal{F}_G$ if*

$$F \in \mathcal{F}_G(U) \quad \Longrightarrow \quad L_X F \in F \cdot (\mathbf{g} \otimes \mathcal{A}(U))\,.$$

(The dot stands for the natural right action).

4 Spinors as odd vector fields

Let (M_0, g_0) be a pseudo Riemannian spin manifold with spinor bundle $S \to M_0$. From now on we will consider the supermanifold $M = M(S) = (M_0, \mathcal{A})$. Functions on M are by definition sections of the exterior algebra $\wedge S$ ($\mathcal{A}(U) = \Gamma(U, \wedge S)$ for any open subset $U \subset M_0$). Let $s^* \in \Gamma(S^*)$ be a section of the dual spinor bundle. Interior multiplication by s^* is an odd derivation $\iota(s^*) : \Gamma(\wedge S) \to \Gamma(\wedge S)$, and hence an odd vector field on M. Choosing a nondegenerate bilinear form $g_1 \in \Gamma(S^* \otimes S^*) \cong \Gamma(Hom(S, S^*))$ we can associate to any spinor field $s \in \Gamma(S)$ the vector field $X_s := \iota(g_1 s)$. Now we can ask:
Does there exist a G-structure $\mathcal{F}_G$ on M such that a spinor field s is a twistor iff X_s is an infinitesimal automorphism of $\mathcal{F}_G$?

In the following I will present the G-structure $\mathcal{F}_G$ which provides a positive answer to this question. The first step is the construction of the linear supergroup $G = \exp \mathbf{g}$ or equivalently of the linear super Lie algebra $\mathbf{g}$

5 Construction of the super Lie algebra g

Definition 6 *A* **(super) extended Poincaré algebra** *of signature (k, l) is a (super) Lie algebra $\mathbf{p}$ with $\mathbb{Z}_2$-grading $\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1$ ($[\mathbf{p}_\alpha, \mathbf{p}_\beta] \subset \mathbf{p}_{\alpha+\beta}$) such that*

i) $\mathbf{p}_0 = Lie\,Isom\,\mathbb{R}^{k,l} = \mathbf{so}(k, l) + \mathbb{R}^{k,l}$ *(Poincaré algebra of signature (k, l)), where $\mathbb{R}^{k,l}$ stands for the pseudo Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$ of signature (k, l).*

ii) $\mathbf{p}_1 = \Sigma =$ *spinor module of* $\mathbf{so}(k, l)$ *trivially extended to a $\mathbf{p}_0$-module and* $[\mathbf{p}_1, \mathbf{p}_1] \subset \mathbb{R}^{k,l}$.

Definition 7 *A bilinear form β on the spinor module is called* **admissible** *if*

1) β is symmetric or skew symmetric. We define the symmetry σ of β to be $\sigma(\beta) = +1$ in the first case and $\sigma(\beta) = -1$ in the second.

2) Clifford multiplication $s \mapsto vs$, $s \in \Sigma$, is either symmetric or skew symmetric for all $v \in \mathbb{R}^{k,l}$. Accordingly, we define the type τ of β to be $\tau(\beta) = \pm 1$.

Let β be admissible. Then the equation

$$(\star) \quad \langle [s,t], v\rangle := \beta(vs,t)\,, \quad s,t \in \Sigma,\ v \in \mathbb{R}^{k,l}\,,$$

defines the bracket $[\cdot,\cdot] : \Sigma \times \Sigma \to \mathbb{R}^{k,l}$ of an extended (respectively super extended) Poincaré algebra if $\sigma(\beta)\tau(\beta) = -1$ (respectively $\sigma(\beta)\tau(\beta) = +1$).

Theorem 1 **[A-C]** *Let $[\cdot,\cdot] : \Sigma \times \Sigma \to \mathbb{R}^{k,l}$ be the Lie bracket (respectively the super Lie bracket) of an extended (respectively super extended) Poincaré algebra. Then $[\cdot,\cdot]$ satisfies equation $(\star)$ for some bilinear form β which is a linear combination of admissible forms β_i such that $\sigma(\beta_i)\tau(\beta_i) = -1$ (respectively $\sigma(\beta_i)\tau(\beta_i) = +1$).*

Put L^- (respectively L^+) := dimension of the vector space of extended (respectively super extended) Poincaré algebras of signature (k,l); $m := k+l$, $s := k-l$. The following remarkable mirror symmetry holds:

Theorem 2 **[A-C]** *L^+ and L^- depend only on m and s modulo 8 and*

$$L^+(-m,s) \quad = \quad L^-(m,s)\,.$$

Now we choose a *nondegenerate* bilinear form β on Σ as in Thm. 1 defining a super extended Poincaré algebra $\mathbf{p}$. Given a super Lie algebra $\mathbf{l}$ the *coadjoint representation* $ad^* : \mathbf{l} \to \mathbf{gl}(\mathbf{l}^*)$, $x \mapsto ad^*_x$, is defined by the equation

$$ad^*_x(y^*) = -(-1)^{\tilde{x}\tilde{y^*}}\, y^* \circ ad_x\,,$$

for homogeneous elements $x \in \mathbf{l}$ and $y \in \mathbf{l}^*$. The coadjoint representation $ad^* : \mathbf{p} \to \mathbf{gl}(\mathbf{p}^*)$ of the super extended Poincaré algebra $\mathbf{p}$ preserves the annihilator $(\mathbb{R}^{k,l} + \Sigma)^\perp$ and hence induces a representation ρ of $\mathbf{p}$ on

$$\mathbf{p}^*/(\mathbb{R}^{k,l}+\Sigma)^\perp \cong (\mathbb{R}^{k,l}+\Sigma)^* \cong \ (\text{via } \langle\cdot,\cdot\rangle \text{ and } \beta)$$

$$\mathbb{R}^{k,l}+\Sigma \cong \mathbb{R}^{m|n}, m = k+l, n = \dim\Sigma.$$

Its image $\mathbf{g} := \rho(\mathbf{p}) \subset \mathbf{gl}_{m|n}(\mathbb{R})$ is a linear super Lie algebra with even part $\mathbf{g}_0 = \mathbf{spin}(k,l)$.

Now we construct a G-structure on the supermanifold $M = M(S)$ for the linear supergroup $G = \exp \mathbf{g}$.

6 The G-structure on M

Consider the $\mathbb{Z}_2$-graded vector bundle $V := TM_0 + S$ of rank $m|n$. The Levi-Civita connection of the pseudo Riemannian manifold (M_0, g_0) induces a canonical covariant derivative ∇ on V. Using its holonomy we can reduce the structure group from $GL_m \times GL_n$ to $Hol(\nabla) \subset Spin(k,l)$. Without restriction of generality we assume that the holonomy group $Hol\,(\nabla) \subset Spin_0(k,l)$ (otherwise it is sufficient to replace G by the linear supergroup generated by G and $Spin(k,l)$). The bilinear form β on the spinor module Σ is $Spin(k,l)$-invariant and hence defines an extension of g_0 to a parallel field $g = g_0 + g_1$ of bilinear forms on V. The data (V, ∇, g_1) immediately provide us with the following *fundamental correspondence*.

Proposition 1 *Any (local) trivialization* (e_i, s_j), $i = 1, \ldots, m$, $j = 1, \ldots, n$, *of the* $\mathbb{Z}_2$*-graded vector bundle* V *defines a (local) frame field* (∇_{e_i}, X_{s_j}) *on* M, $X_s = \iota(g_1 s)$.

Let (e_i, s_j) be a trivilization of $V|_U$ which belongs to the holonomy reduction, $U \subset M_0$ open. Denote by $F = (\nabla_{e_i}, X_{s_j}) \in \mathcal{F}(U)$ the corresponding local frame field on M. Now the linear supergroup $G = \exp \mathbf{g}$ associated to the the linear super Lie algebra $\mathbf{g}$ constructed in the previous section comes into play. The orbit

$$\mathcal{F}_G(U) = F \cdot G(M|_U)$$

is independent of the trivialization of $V|_U$ (as long as the trivialization belongs to the holonomy reduction) since $\mathbf{g} \supset \mathbf{spin}(k,l) = \mathbf{g}_0$. Moreover $\mathcal{F}_G$ is a G-structure on M.

Theorem 3 **[A-C-D-S]** $s \in \Gamma(S)$ *is a twistor iff* X_s *is an infinitesimal automorphism of* $\mathcal{F}_G$.

Proof: We compute:

$$L_{X_s} F = (-X_{\nabla_{e_i} s}, 0) = F \cdot \begin{pmatrix} 0 & 0 \\ \tilde{C}(s) & 0 \end{pmatrix}, \quad \tilde{C}(s) = -(s^j(\nabla_{e_i} s))_{ij} \,,$$

where (s^j) is the basis dual to (s_j). This shows that X_s is an infinitesimal automorphism iff

$$\begin{pmatrix} 0 & 0 \\ \tilde{C}(s) & 0 \end{pmatrix} \in \mathbf{g}_1 \otimes C^\infty(U).$$

On the other hand, the explicit description of the representation ρ defining $\mathbf{g} = \rho(\mathbf{p})$ gives:

$$\mathbf{g}_1 \otimes C^\infty(U) = \left\{ \begin{pmatrix} 0 & 0 \\ C(\tilde{s}) & 0 \end{pmatrix} | C(\tilde{s}) = -(s^j(e_i \tilde{s}))_{ij} \,, \quad \tilde{s} \in \Gamma(S) \right\},$$

Therefore we obtain the equation $\tilde{C}(s) = C(\tilde{s})$ or, more explicitely,

$$\nabla_{e_i} s = e_i \tilde{s}, \ i = 1, \ldots, m.$$

This is equivalent to the twistor equation.

References

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[2] D.V. Alekseevsky, V. Cortés, C. Devchand, U. Semmelmann: *Killing spinors are Killing vector fields in Riemannian supergeometry*, Journal of Geometry and Physics **25 (3)** (1997), 14p.

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