The Rigid Body with Two Controls and the Brockett Perturbation

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Abstract

The perturbation of the rigid body equations with two controls by a dissipation of Brockett type is discussed and some of its properties are pointed out.

1 Introduction

It is well known that a lot of dynamical systems have a Lax formulation, i.e. their dynamics can be put in the equivalent form

$$\dot{L} = [L, B], \tag{1.1}$$

where L, B are matrices and $[L, B] = LB - BL$ is their commutator. For instance this is the case of the 1-dimensional harmonic oscillator, the free rigid body, the Maxwell-Bloch equations from laser matter dynamics, Toda lattice and so on. If we add now to the equation (1.1) a Brockett perturbation of the form

$$[L, [L, B]],$$

then (1.1) becomes

$$\dot{L} = [L, B] + [L, [L, B]]. \tag{1.2}$$

It is a very tempting problem to determine the connection between the dynamics of (1.1) and that of (1.2). The problem is still open and are known only some partial results and examples([1]).

The goal of our paper is to give an answer to this problem for the particular case of the rigid body with two controls. As a particular case which will be mentioned later we find again one of the examples from ([1]).

2 The rigid body equations with two controls

The rigid body equations with two controls about the minor and the major axis are given by:

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3 + u_1, \\ \dot{m}_2 = a_2 m_1 m_3, \\ \dot{m}_3 = a_3 m_1 m_2 + u_3, \end{cases} \tag{2.1}$$

where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2}, a_2 = \frac{1}{I_1} - \frac{1}{I_3}, a_3 = \frac{1}{I_2} - \frac{1}{I_1},$$

I_1, I_2, I_3 being the principal moments of inertia of the body and we suppose as usually that

$$I_1 > I_2 > I_3 > 0.$$

Now we employ the feedbacks:

$$u_1 = -km_2m_3, \quad u_3 = km_1m_2, \tag{2.2}$$

where $k \in \mathbf{R}$ is the feedback gain parameter or the controlled parameter.

Theorem 2.1 ([3]). *The controlled system (2.1), (2.2) has a Lax formulation,i.e.*

$$\dot{M} = [M, B],$$

where

$$M = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -m_3/I_3 & m_2/I_2g \\ m_3/I_3 & 0 & -m_1/I_1 \\ -m_2/I_2g & m_1/I_1 & 0 \end{bmatrix}$$

and

$$g = \frac{1}{1 + kI_2}.$$

Theorem 2.2 ([1]). *The functions H and C given respectively by:*

$$H(m_1, m_2, m_3) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2g} + \frac{m_3^2}{I_3}\right) \tag{2.3}$$

$$C(m_1, m_2, m_3) = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) \tag{2.4}$$

are constants of the motion.

It is not hard to see that the controlled system (2.1),(2.2) has the following equilibrum states

$$e_1 = (M, 0, 0), \quad e_2 = (0, M, 0), \quad e_3 = (0, 0, M), \tag{2.5}$$

where $M \in \mathbf{R}$. Then we can prove:

Theorem 2.3 *For the equilibrium states e_1, e_2, e_3 we have the following behaviours:*
(i) If $k \in (-\infty, -a_3)$, then e_1 is spectrally unstable;
(ii) If $k \in (-a_3, a_1)$, then e_2 is spectrally unstable;
(iii) If $k \in (a_1, \infty)$, then e_3 is spectrally unstable.

Proof. (i) The linear part of the controlled system (2.1), (2.2) at the equilibrium state e_1 has the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_2M \\ 0 & (a_3 + k)M & 0 \end{bmatrix}.$$

Its characteristic polynomial has the eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm M\sqrt{a_2(a_3+k)}$$

and then e_1 is spectrally unstable if $k \in (-\infty, -a_3)$.
(ii) The linear part of the controlled system (2.1), (2.2) at the equilibrium state e_2 has the following matrix:

$$\begin{bmatrix} 0 & 0 & (a_1-k)M \\ 0 & 0 & 0 \\ (a_3+k)M & 0 & 0 \end{bmatrix}.$$

Its characteristic polynomial has the eigenvalues $\lambda_1 = 0,\ \lambda_{2,3} = \pm M\sqrt{(a_1-k)(a_3+k)}$, so that for $k \in (-a_3, a_1)$ the equilibrium state e_2 is spectrally unstable.
(iii) The linear part of the controlled system (2.1), (2.2) at the equilibrium state e_3 has the following matrix:

$$\begin{bmatrix} 0 & a_1-k & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its characteristic polynomial has the eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm M\sqrt{a_2(a_1-k)},$$

so that for $k \in (a_1, \infty)$ the equilibrium state e_3 is spectrally unstable (q.e.d.).

3 Controlled system and Brockett perturbation

Let us add now to the controlled system (2.1), (2.2) a Brockett perturbation. It takes the following form:

(3.1) $$\dot{M} = [M, B] + [M, [M, B]]$$

or equivalently

$$\begin{aligned} \dot{m}_1 &= (a_1-k)m_2m_3 + (a_3+k)m_1m_2^2 - a_2m_1m_3^2, \\ \dot{m}_2 &= a_2m_1m_3 + (a_1-k)m_2m_3^2 - (a_3+k)m_2m_1^2, \\ \dot{m}_3 &= (a_3+k)m_1m_2 + a_2m_3m_1^2 - (a_1-k)m_3m_2^2. \end{aligned}$$

Theorem 3.1 *(i) The function C given by (2.4) is a constant of the motion for the system (3.1).*
(ii) For H given by (2.3) we have

$$\frac{dH}{dt} \leq 0$$

along the trajectories of (3.1). Moreover

$$\frac{dH}{dt}(m_1, m_2, m_3) = 0$$

if and only if $m_1 = m_2 = m_3 = 0$.

Proof. (i) Indeed, we have

$$\frac{dC}{dt} = m_1\dot{m}_1 + m_2\dot{m}_2 + m_3\dot{m}_3 = 0.$$

(ii) A short computation shows that

$$\frac{dH}{dt} = -[(a_1 - k)^2 m_2^2 m_3^2 + a_2^2 m_1^2 m_3^2 + (a_3 + k) m_1^2 m_2^2]$$

and then our assertion follows easily (q.e.d.).

Theorem 3.2 *The equilibrium states (2.5) are also equilibrium states of the system (3.1) and moreover we have:*
(i) If $k \in (-\infty, -a_3)$, *then* e_1 *is spectrally unstable;*
(ii) If $k \in (-a_3, a_1)$, *then* e_2 *is spectrally unstable;*
(iii) If $k \in (a_1, \infty)$, *then* e_3 *is spectrally unstable.*

Proof. Without loss of generality we can suppose that $M = 1$.
(i) The characteristic polynomial of the linear part of our system (3.1) at the equilibrium state e_1 is given by

$$\lambda[\lambda^2 + (a_3 - a_2 + k)\lambda - 2a_2(a_3 + k)].$$

Its nonzero roots are of opposite sign since

$$\Delta = (a_3 - a_2 + k)^2 + (a_3 + k)a_2 \geq 0$$

and

$$P = -2(a_3 + k)a_2 < 0.$$

It follows that e_1 is spectrally unstable.
(ii) The caracteristic polynomial of the linear part of our system (3.1) at the equilibrium state e_2 is given by

$$\lambda\{\lambda^2 + [(a_1 - k) - (a_3 + k)]\lambda - 2(a_1 - k)(a_3 + k)\}.$$

Its nonzero roots are of opposite sign since

$$\Delta = [(a_1 - k) - (a_3 + k)]^2 + 4(a_1 - k)(a_3 + k) \geq 0$$

and

$$P = -2(a_1 - k)(a_3 + k) < 0.$$

It follows that e_2 is spectrally unstable.
(iii) The characteristic polynomial of the linear part of our system(3.1) at the equilibrium state e_3 is given by

$$\lambda[\lambda^2 + (a_2 - a_1 + k)\lambda - 2a_2(a_1 - k)].$$

Its nonzero roots are of opposite sign since

$$\Delta = (a_2 + a_1 - k)^2 + 2a_2(a_1 - k) \geq 0$$

and

$$P = -a_2(a_1 - k) < 0.$$

It follows that the equilibrium state e_3 is spectrally unstable (q.e.d.).
Remark 3.1 For the particular case $k = 0$ we find again some of the results given in [1].

4 Conclusions

In this paper we have discussed some connections between the dynamics (2.1), (2.2) and (3.1). In the particular case $k = 0$ we find again the results of Bloch, Krishnaprasad, Marsden and Raţiu [1] for the free rigid body.
Let us finish with the remark that it is an open problem to determine the relation between the nonlinear stability of the equilibrium states (2.5) for the systems (2.1), (2.2) and respectively (3.1).

References

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