On the Generalized Lagrange Spaces Associated to the Remarkable Lagrangians

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Abstract

One considers a Lagrange space with the fundamental function (1). For the generalized Lagrange space for relativistic optics, GL^n, one determines the nonlinear connection, the canonical metrical connection together with curvatures and torsions, Einstein equations.This is the summary of a part of our paper [2].

Let M be a real, n-dimensional C^∞ manifold and (TM, π, M) its tangent bundle and (x^i, y^i), $(i, j, k, ... = 1, 2, ..., n)$ the canonical coordinates of the points $u \in TM$, $\pi(u) = x$, in a coordinate neighbourhood $\pi^{-1}(U)$, where U is a coordinate neighbourhood of M at x, i.e. $x \in U \subset M$. One considers a Riemanian metric $\gamma_{ij}(x)$ on M and the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

We shall consider the real function L on TM:

$$L(x,y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{mc}A_i(x)y^i + \frac{4e^2}{m^2c^2}(A_i(x)y^i)^2, \tag{1}$$

where $A_i(x)$ is a covector field defined on M, or on an open set of M. We set

$$a = \frac{2e}{mc}. \tag{2}$$

Proposition 1 *The fundamental tensor field of the Lagrange space $L^n = (M, L(x, y))$ is*

$$g_{ij}(x) = mc\gamma_{ij}(x) + a^2 A_i(x)A_j(x). \tag{3}$$

Theorem 2 *The following properties hold:*
1^0 *The d-tensor field g_{ij} is symmetric, non-degenerate and globally defined on M .*
2^0 *The contravariant tensor of (g_{ij}) is as follows:*

$$g^{ij}(x) = \frac{1}{mc}\gamma^{ij}(x) - \frac{1}{1+d^2}A^i(x)A^j(x), \tag{4}$$

where

$$\begin{cases} A^i(x) = \frac{a}{mc}\gamma^{ij}(x)A_j(x), \\ d^2 = \frac{a^2}{mc}\gamma^{ij}(x)A_i(x)A_j(x). \end{cases} \tag{5}$$

Theorem 3
1^0 *The pair $L^n = (M, L(x, y))$, where $L(x, y)$ is given by (4), is a Lagrange space.*
2^0 *The pair $R^n = (M, g_{ij}(x))$ is a Riemannian space.*

Let us consider the d-tensor field

$$\tilde{g}_{ij}(x,y) = mc\gamma_{ij}(x) + a^2\tilde{y}_i\tilde{y}_j. \tag{6}$$

where

$$\tilde{y}_i = \frac{mc}{a}\gamma_{ij}(x)y^j. \tag{7}$$

Taking into account the section $S_A : M \to TM$, defined by

$$S_A : \begin{cases} x^i = x^i \\ y^i = A^i(x) \end{cases} \tag{8}$$

we see that $\tilde{g}_{ij}|S_A = g_{ij}$ from (3).

Theorem 4

1^0 *The pair $GL^n = (M, \tilde{g}_{ij})$ is a generalized Lagrange space.*

2^0 *The restriction of GL^n to the section S_A is the Riemannian space $R^n = (M, g_{ij}(x))$.*

In this case, the square of the norm of y is

$$||y||_1^2 = \tilde{y}_j\tilde{y}^j = \frac{mc}{a}\gamma_{ij}(x)y^iy^j. \tag{9}$$

Theorem 5 *The generalized Lagrange space GL^n is not reducible to a Lagrange space.*

We define the following functions on TM:

$$a_\sigma(x,y) = mc(1 + a\sigma||y||_1^2), \sigma \in N. \tag{10}$$

Proposition 6 *The coefficients C^i_{jk} of the vertical part of the canonical metrical d-connection of GL^n are*

$$C^i_{jk} = \frac{m^2c^2}{a_1}\gamma_{jk}y^i. \tag{11}$$

Let $\mathcal{E}(x,y)$ be the absolute energy of the metric $\tilde{g}_{ij}(x,y)$:

$$\mathcal{E}(x,y) = \tilde{g}_{ij}y^iy^j = a\frac{a_1}{mc}||y||_1^2. \tag{12}$$

The absolute energy $\mathcal{E}(x,y)$ is globally defined on TM.

Theorem 7 *The space $\mathcal{E}^n = (M, \mathcal{E}(x,y))$ is a Lagrange space.*

Theorem 8 *The geodesics of the Lagrange space $\mathcal{E}^n = (M, \mathcal{E}(x,y))$ have the same equations as that of the Riemannian space $V^n = (M, \gamma_{ij}(x))$.*

The metric $\tilde{g}_{ij}(x,y)$ of the generalized Lagrange space GL^n being weakly regular, the function $N^{*i}_j = \frac{\partial G^{*i}}{\partial y^j}$ gives us

$$N^{*i}_j(x,y) = \{^{\ i}_{jk}\}y^k. \tag{13}$$

Proposition 9 *The Berwald connection of the generalized Lagrange space GL^n has the coefficients $(\{^{\ i}_{jk}\}y^k, \{^{\ i}_{jk}\}, 0)$.*

Proposition 10 *The horizontal curves in generalized Lagrange spaces GL^n, with respect to the nonlinear connection N^*, are given by the system of differential equations*

$$\frac{dy^i}{dt} + \{^{\ i}_{jk}\}y^j\frac{dx^k}{dt} = 0. \tag{14}$$

Proposition 11 *The h-geodesics, with respect to the nonlinear connection N^*, of general Lagrange spaces GL^n coincide with the geodesics of the Lagrange space $\mathcal{E}^n$.*

Theorem 12 *The coefficients L^i_{jk} of the canonical metrical d-connection $L\Gamma(N^*)$ of the generalized Lagrange space GL^n coincide with the coefficients of the Levi-Civita connection of the Riemann space $V^n = (M, \gamma_{ij}(x))$:*

$$L^i_{jk} = \{^{\,i}_{jk}\}. \tag{15}$$

Proposition 13 *The h-paths of the canonical metrical d-connection $L\Gamma(N^*)$ have the same equation as the geodesics of the Riemannian space V^n.*

Proposition 14 *The v-paths of the canonical metrical d-connection $L\Gamma(N^*)$, in a point $x_0 \in M$, are given by*

$$\frac{d^2y^i}{dt^2} + \frac{m^2c^2}{a_1}\gamma_{jk}(x_0)y^i\frac{dy^i}{dt}\frac{y^k}{dt} = 0. \tag{16}$$

Theorem 15 *The Einstein equations of generalized Lagrange spaces $GL^n(n > 2)$ are given by*

$$\frac{1}{mc}(r_{ij} - \frac{1}{2}r\gamma_{ij}) + t_{ij} = \chi\overset{H}{T}_{ij}\ , \tag{17}$$

$$\frac{m^2c^2}{a_1}\tilde{g}^{rs}\{(\gamma_{ij}\gamma_{rs} - \gamma_{ir}\gamma_{js}) - \frac{1}{2}\tilde{g}^{pq}(\gamma_{pq}\gamma_{rs} - \gamma_{pr}\gamma_{qs})\tilde{g}_{ij}\} = \chi\overset{V}{T}_{ij}\ , \tag{18}$$

$$t_{ij} = \frac{1}{a_1}(\tilde{g}_{ij}r_{oo} - \frac{a_1a^2}{2m^2c^2}ry_iy_j - mcr_{iojo}). \tag{19}$$

References

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