An L^∞ *a Priori* Estimate for the Calabi-Yau Operator on Compact almost-Kähler Manifolds

Ph. Delanoë[1]

Abstract

In this note, we recall the definition and main properties of an almost-Kähler analogue of the Calabi-Yau operator, and we carry out for it an L^∞ *a priori* estimate which extends the one obtained by Yau and Aubin on compact Kähler manifolds.

1 The Calabi-Yau operator on compact almost-Kähler manifolds

Let (M, J, σ_o) be a compact manifold of even dimension $2m$ with an almost-complex structure J *tamed* by a symplectic form σ_o. The latter means that,

$$\sigma_o(J., J.) \equiv \sigma_o,$$

and also that the tensor $\sigma_o(., J.)$, which is *symmetric*, is furthermore **positive-definite** (whenever a 2-form fulfills this positivity condition, we say that it is *J-positive*). If J is integrable, the manifold is called Kähler, and if not, almost-Kähler.
Gromov pointed out that on any compact almost-Kähler manifold (M, J, σ_o), one can consider an analogue of the Calabi-Yau operator. Let us recall its definition and main properties [2]. First of all, let us call a smooth real function ϕ on M, *admissible*, provided the cohomologous 2-form

$$\sigma_\phi = \sigma_o + dJ^t d\phi$$

remains *J-positive* on M (where J^t stands for the *transposed* endomorphism field). Consider the two sets, both *convex*:

$$\begin{aligned} A &= \{\phi \in C^\infty(M), \int_M \phi\sigma_o^m = 0, \sigma_\phi \text{ is J-positive}\}, \\ B &= \{f \in C^\infty(M), \int_M f\sigma_o^m = \int_M \sigma_o^m, f > 0\}. \end{aligned}$$

Then the operator F, defined by the formula

$$F(\phi)\sigma_o^m := \sigma_\phi^m,$$

sends A into B. The almost-complex structure J yields a *splitting* of the complexified tangent bundle; accordingly, the 2-form σ_ϕ splits into a $(2, 0)$-part, here denoted by τ_ϕ,

[1] Chargé de recherches au CNRS

a real $(1,1)$-part denoted by H_ϕ and a $(0,2)$-part which is the complex conjugate $\overline{\tau_\phi}$ of its $(2,0)$-part (since σ_ϕ is *real*). Now the operator F splits into

$$F = F_o + \ldots + F_{[m/2]},$$

(where $[m/2]$ stands for the *integer part* of $m/2$), with each F_j defined by

$$F_j \sigma_o^m := \frac{m!}{(j!)^2(m-2j)!} \tau^j \wedge \overline{\tau}^j \wedge H^{m-2j}.$$

The differential operator $\phi \to \tau_\phi$ is linear with respect to the gradient of ϕ and to the *torsion* of J. So, in the Kähler case, it identically vanishes and $F \equiv F_o$. The main result which we have obtained [2] on F is the following

Theorem 1 *The operator $F : A \to B$ is elliptic, one-to-one, locally invertible, hence a diffeomorphism onto its image $F(A)$. Moreover, when $m = 2$, the image $F(A)$ coincides with all of the target-set B if and only if the torsion of J vanishes (Kähler case).*

The "if" part of the final statement follows from the Calabi-Yau theorem [4] (combined with the C^∞ version of the Newlander-Nirenberg theorem [3]). The "only if" part relies on the idea of constructing a function ϕ which lies on *the boundary* of the source-set A with its image $F(\phi)$ able to lie *inside* the target-set B due to the non-vanishing of the torsion. We have conjectured in [2] that the final statement remains valid in higher dimension.

2 An L^∞ *a priori* estimate

From now on, we assume that the almost-complex structure J is *not* integrable. In this setting, we shall extend for the operator F Yau's main estimate [4] (following Aubin's device [1]).

Lemma 2 *For any non-decreasing C^1 function $p : \mathbb{R} \to \mathbb{R}$ and for any $\phi \in A$, the following inequality holds*

$$\frac{1}{2m} \int_M p'(\phi) |d\phi|^2 \sigma_o^m \le \int_M p(\phi)(\sigma_o^m - \sigma_\phi^m).$$

Here $|d\phi|$ stands for the norm of $d\phi$ in the hermitian metric

$$h_o(.,.) := \sigma_o(.,J.).$$

Proof. Let us fix p and ϕ as stated, and compute

$$\sigma_o^m - \sigma_\phi^m = (\sigma_o - \sigma_\phi) \wedge \sum_{k=0}^{m-1} (\sigma_o^{m-1-k} \wedge \sigma_\phi^k),$$

which readily holds due to the commutativity of the exterior product of 2-forms. Recalling the definition of σ_ϕ, we get

$$\int_M p(\phi)(\sigma_o^m - \sigma_\phi^m) = \frac{1}{2}\int_M p(\phi) dJ^t d\phi \wedge \sum_{k=0}^{m-1}(\sigma_o^{m-1-k} \wedge \sigma_\phi^k).$$

Now we use Stoke's theorem on all of M, where both σ_o and σ_ϕ are *closed* forms. Since M has no boundary, it yields

$$\begin{aligned}\int_M p(\phi)(\sigma_o^m - \sigma_\phi^m) &= -\frac{1}{2}\int_M p'(\phi) d\phi \wedge (J^t d\phi) \wedge \sum_{k=0}^{m-1}(\sigma_o^{m-1-k} \wedge \sigma_\phi^k) \\ &= \frac{1}{2}\int_M p'(\phi)(J^t d\phi) \wedge d\phi \wedge \sigma_o^{m-1} \\ &\quad + \sum_{k=0}^{m-1}\int_M p'(\phi)(J^t d\phi) \wedge d\phi \wedge \sigma_o^{m-1-k} \wedge \sigma_\phi^k.\end{aligned}$$

Applying the positivity lemma of [2, appendix 3] , we obtain

$$\int_M p(\phi)(\sigma_o^m - \sigma_\phi^m) \geq \frac{1}{2}\int_M p'(\phi)(J^t d\phi) \wedge d\phi \wedge \sigma_o^{m-1} \equiv \frac{1}{2m}\int_M p'(\phi)|d\phi|^2 \sigma_o^m$$

hence the desired result $\square$

With this lemma at hand, we can follow Aubin-Kazdan's device [1, pp. 148–149] and consider, for any integer $n \geq 2$, the function

$$p(\phi) = \phi|\phi|^{n-2}$$

which satisfies:

$$p'(\phi) = (n-1)|\phi|^{n-2}.$$

From the preceding lemma and the definition of F, we get from this choice of p,

$$\int_M |\phi|^{n-2}|d\phi|^2 d\mu \leq \frac{2M}{(n-1)}\int_M \phi|\phi|^{n-2}[1 - F(\phi)] d\mu \tag{1}$$

where $d\mu$ stands for the Lebesgue measure associated to the hermitian metric h_o. Note that the following identity holds $d\mu$-almost everywhere on M:

$$|\phi|^{n-2}|d\phi|^2 \equiv |d(|\phi|^{\frac{n}{2}})|^2.$$

Eventually, for any $\phi \in A$, we get

$$\int_M |d(|\phi|^{\frac{n}{2}})|^2 d\mu \leq \frac{mn^2}{2(n-1)}\int_M \phi|\phi|^{n-2}[1 - F(\phi)] d\mu$$

which is analogous to the estimate (15) of [1, p. 148].

To proceed further, we note that the J-positivity of σ_ϕ implies the existence of a positive constant α such that, for any $\phi \in A$:

$$\Delta\phi > -2m - \alpha|d\phi|$$

(here Δ stands for the Laplacian in the metric h_o). Since $\int_M (\Delta\phi) d\mu = 0$, we infer an L^1 bound:

$$\int_M |\Delta\phi| d\mu \leq 4mV + 2\alpha \int_M |d\phi| d\mu, \tag{2}$$

where V stands for the *volume* of $(M, d\mu)$. Moreover, using a positive Green function of the Laplacian (*cf.* [1, p. 108]), we get a constant β such that, for any function ϕ on M with zero $d\mu$-average:

$$\int_M |\phi| d\mu \leq \beta \int_M |\Delta\phi| d\mu.$$

This, together with (2), yields

$$\int_M |\phi| d\mu \leq 4mV\beta + 2\alpha\beta \int_M |d\phi| d\mu$$

which in turn, combined with (1) read with $n = 2$, implies

$$\int_M |d\phi|^2 d\mu \leq 4m\beta \left(2mV + \alpha \int_M |d\phi| d\mu\right) \left[1 + \|F(\phi)\|_{L^\infty}\right].$$

Using Schwarz inequality

$$\int_M |d\phi| d\mu \leq \sqrt{V} \left(\int_M |d\phi|^2 d\mu\right)^{\frac{1}{2}},$$

we finally get

Proposition 3 *There exists a constant $C = C(m, V, \alpha, \beta)$ such that, for any $\phi \in A$,*

$$\|d\phi\|_{L^2} \leq C \left[1 + \|F(\phi)\|_{L^\infty}\right].$$

From this and the Poincaré inequality, we infer

$$\|\phi\|_{L^2} \leq \frac{C}{\sqrt{\lambda_1}} \left[1 + \|F(\phi)\|_{L^\infty}\right]$$

(where λ_1 stands for the first positive eigenvalue of Δ). Now one can perform the same induction on $\|\phi\|_{L^n}$, for $n \in \mathbb{N}, n \geq 2$, as the one in [1, pp. 148–149] (which makes use of a Sobolev inequality). It yields a bound on $\|\phi\|_{L^\infty}$ in terms of $(m, V, \alpha, \beta, \lambda_1, \|F(\phi)\|_{L^\infty})$, valid for all $\phi \in A$.

The author wishes to thank the organizers of the Sibiu meeting for their invitation and kind hospitality.

References

[1] Thierry Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Grundlehren der mathematischen Wissenschaften 252, Springer-Verlag New-York (1982).

[2] Ph. Delanoë, *Sur l'analogue presque-complexe de l'équation de Calabi-Yau*, Osaka J. Math. 33 (1996), 829-846.

[3] A. Nijenhuis & W.B. Woolf, *Some integration problems in almost-complex and complex manifolds*, Annals of Math. 77 (1963), 424-489.

[4] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. XXXI (1978), 339-411.

Université de Nice-Sophia-Antipolis
Mathématiques, Parc Valrose
F-06108 NICE CEDEX 2
e-mail: delphi@math.unice.fr