Conformal Scalar Curvature on the Hyperbolic Space

Erwann Delay[1]

Abstract

Regarding the theme of the conformal scalar curvature on the hyperbolic space, we bring here a study of the fine asymptotic behaviour in any dimension. We always deal with general semi-linear equations, before applying our results to the particular case of the geometric equation.

1 Preliminary

Let B the unit ball in $\mathbb{R}^n$ (with the Euclidean metric E) and let ρ the function defined on B by:

$$\rho(x) = \frac{1}{2}(1 - |x|^2).$$

The model of hyperbolic space $\mathbb{H}^n(-1)$ chosen is B with the conformal metric :

$$H_0 = \rho^{-2}E.$$

For $v > -1$ real function on B, we define the *conformal metric*

$$H_v = (1 + v)H_0.$$

We are interested in the *scalar curvature* of such conformal metrics. We are going to show that the map

$$v \longrightarrow Scal(H_v) - Scal(H_0)$$

is invertible near zero in some appropriate spaces. Let u in $C^{k,\alpha}_{loc}(B)$; we will say that u is in $\Lambda^s_{k,\alpha}$ if the next quantity, who represents his norm on this space, is *finite*:

$$\| u \|^s_{k,\alpha} := \sum_{|\gamma| \le k} \sup_{x \in B} [\rho(x)^{-s+|\gamma|} |\partial^\gamma u(x)|] + \sum_{|\gamma| = k} \sup_{\substack{x,y \in B \\ x \neq y}} \min(\rho(x)^{-s+k+\alpha}, \rho(y)^{-s+k+\alpha}) \frac{|\partial^\gamma u(x) - \partial^\gamma u(y)|}{|x - y|^\alpha}.$$

According to [7], if $K \in \Lambda^0_{k,\alpha}$ verifies

$$\inf_B K > s[s - (n - 1)]$$

then the operator of Schrödinger type :

$$\triangle + K : \Lambda^s_{k+2,\alpha} \longrightarrow \Lambda^s_{k,\alpha}$$

[1] Moniteur-Allocataire MENESR

is an *isomorphism.*
Besides, the linearisation of the map $v \longrightarrow Scal(H_v)$ at zero is given by

$$d_v(Scal(H_v))(0)\delta v = (n-1)(\triangle + n)\delta v.$$

Thus by the inverse function theorem, if $0 \le s < n$, the map

$$v \longrightarrow Scal(H_v) - Scal(H_0)$$

from $\Lambda^s_{k+2,\alpha}$ to $\Lambda^s_{k,\alpha}$ is invertible near zero.

Note that according to [10], if $s > n$, there is no metric H on B satisfying

$$H - H_0 \in \Lambda^{s-2}_2 \text{ and } Scal(H) \ge Scal(H_0)$$

different of H_0.

Now we are going to specify further the solvability of the equations like that of the conformal scalar curvature. Details can be found in [5].

2 Conformal scalar curvature in dimension $n \ge 3$.

Let's consider the conformal metric (for $v > -1$ real function on B)

$$H = (1+v)^{\frac{4}{n-2}} H_0.$$

Then its scalar curvature is (e.g [A], p. 126):

$$Scal(H) = (1+v)^{-\frac{n+2}{n-2}}[4\frac{n-1}{n-2}\triangle v + S_0(1+v)],$$

where $S_0 = Scal(H_0)$=-n(n-1). To prescribe $Scal(H) = S_0 + \sigma$, we study the more general equation :

$$\triangle v = A(1+v) + (g-A)(1+v)^p, \qquad (E)$$

with $p > 1$ and $A - g \ge 0$. Results for $Scal(H)$ will be infered from the case $p = \frac{n+2}{n-2}$, $A = \frac{n(n-2)}{4}$ and $g = \frac{n-2}{4(n-1)}\sigma$.
We rewrite the equation (E) in the form :

$$\triangle v + A(p-1)v = \underbrace{A(1+v) + (g-A)(1+v)^p + A(p-1)v}_{\Phi(x,v)} .$$

To solve it, we will construct upper and lower solutions.
When s satisfies :

$$\text{(C)} \begin{cases} A(p-1) > s(s-(n-1)) & \text{if} \quad s < 0 \text{ or } s \ge \frac{n-2}{2} , \\ A(p-1) > -\frac{sn}{2} & \text{if} \quad 0 \le s \le \frac{n-2}{2} , \end{cases}$$

we take

$$M := \begin{cases} \frac{1}{A(p-1)+s(n-1-s)} & \text{if} \quad s<0 \text{ or } s \geq \frac{n-2}{2} \\ \frac{1}{A(p-1)+\frac{sn}{2}} & \text{if} \quad 0 \leq s \leq \frac{n-2}{2} \end{cases} .$$

Then the function $\varphi = M\rho^s$ satisfies

$$(\triangle + A(p-1))\varphi \geq \rho^s.$$

So we can give the

Proposition 1 *If g is in $\Lambda_0^s \bigcap C_{loc}^{k,\alpha}(B)$, $s \geq 0$ satisfies (C), $0<\alpha<1$ and if there exist $\lambda \geq 0$ and $\delta \in]0,1[$ so that, if we set $q = \frac{1-\delta^p}{1-\delta}$ (then $q \in]1,p[$), we have:*

$$(1) \qquad -(1-\delta)[\frac{1}{M} - A(p-1) + q(A-g) - A] \leq (2\rho)^{-s}(g) \leq \lambda(\frac{1}{M} - pg),$$

then there exists a solution $v > -1$ of (E) in $\Lambda_0^s \bigcap C_{loc}^{k+2,\alpha}$.

Proof. We show that $v^+ = \frac{2^s\lambda\varphi}{M}$ is an upper-solution and $v^- = \frac{-2^s(1-\delta)\varphi}{M}$ is a lower-solution using the convexity of the function $t \longrightarrow (1+t)^p$ on $t>-1$ for $p>1$.
By the maximum principle (cf. [6], p. 33, th. 3.3), we have the

Proposition 2 *If g is in $C^0(B)$, $(A-g) \geq 0$, v is a solution of (E) in $C_{loc}^2(B) \bigcap C^0(\bar{B})$, $v > -1$ and $v=0$ on ∂B, then v is unique.*

Note that our preceding solution is unique as soon as $s>0$.
Can we have a better weighted regularity for it when $g \in \Lambda_{k,\alpha}^s$? First we have

Lemma 1 *Assume $g \in \Lambda_{k,\alpha}^s$, $s \geq 0$ satisfies the condition (C) and $0<\alpha<1$. Suppose that:*

$$(2) \qquad A > \frac{n-1+s(s-(n-1))}{p-1} \; ,$$

$$(3) \qquad \inf_{x\in B}[(A-g)p(1-(1-\delta)(2\rho)^s)^{p-1} - A - (n-1)] > s(s-(n-1)) \; .$$

If v is a solution of (E) in $\Lambda_{k+2,\alpha}^s$ with $-(1-\delta)(2\rho)^s \leq v \leq \lambda(2\rho)^s$, then $\| v \|_{k+2,\alpha}^s \leq C$ where C is independent of v (which we denote by $C = C(\hat{v})$).

Proof. We know that

$$\| v \|_0^s \leq 2^s \max((1-\delta),\lambda) \| \rho^s \|_0^s = 2^s \max((1-\delta),\lambda) = C(\hat{v}).$$

We will show that $\| \nabla v \|_0^{s-1} \leq C(\hat{v})$. Then $\| v \|_1^s \leq C(\hat{v})$, thus $\forall \alpha \in]0,1[$,

$$\| v \|_{0,\alpha}^s \leq C(\hat{v}).$$

We differentiate the equation (E) :

$$\triangle\nabla_i v + \underbrace{[(A-g)p(1+v)^{p-1} - A - (n-1)]}_{K(x)} \nabla_i v = (1+v)^p \nabla_i g.$$

We have $\inf_{x\in B} K(x) > s(s-(n-1))$; then according to [7],

$$\| \nabla v \|_0^{s-1} \leq C(\hat{v}) \| (1+v)^p \nabla g \|_0^{s-1} \leq C(\hat{v}).$$

By induction, we have (by estimation of Schauder type)

$$\| v \|_{l+2,\alpha}^s \leq C(\hat{v}).$$

Proposition 3 *We suppose : $g \in \Lambda_{k,\alpha}^s$, $s > 0$ verify (C), $g \leq A$,*

$$A > \frac{(n-1)+s(s-(n-1))}{p-1}, \tag{2}$$

and:

$$\inf_{x\in B}[(A-|g|)p(1-(1-\delta)(2\rho)^s)^{p-1} - A - (n-1)] > s(s-(n-1)) \ , \tag{4}$$

$$-(1-\delta)[\frac{1}{M} - A(p-1) + q(A-|g|) - A] \leq (2\rho)^{-s}|g| \leq \lambda(\frac{1}{M} - p|g|) \ . \tag{5}$$

Then the unique solution $v \in \Lambda_0^s \bigcap C_{loc}^{k+2,\alpha}$ of the equation (E) with $v > -1$ lies in fact in $\Lambda_{k+2,\alpha}^s$.

Proof. We use a continuity method. We consider for $t \in [0,1]$ the family of equations:

$$\triangle v = (tg - A)(1+v)^p + A(1+v), \ v > -1 \tag{E$_t$}$$

and the set $\mathcal{S} = \{t \in [0,1], \exists v_t \in \Lambda_{k+2,\alpha}^s \text{ solution of } (\mathrm{E}_t) \text{ with } v_t > -1\}$.
a) $\mathcal{S} \neq \phi$ because $0 \in \mathcal{S}$ (we can take $v_0 = 0$).
b) $\mathcal{S}$ is open in $[0,1]$:
We use $\Psi : [0,1] \times \Lambda_{k+2,\alpha}^s \longrightarrow \Lambda_{k,\alpha}^s$ defined by :

$$\Psi(t,v) = \triangle v - (tg - A)(1+v)^p - A(1+v)$$

and we show that $\frac{\partial\Psi}{\partial v}(t_0, v_0)$ is an isomorphism from $\Lambda_{k+2,\alpha}^s$ to $\Lambda_{k,\alpha}^s$ for all $(t_0, v_0) \in [0,1] \times \Lambda_{k+2,\alpha}^s$ solution of $\Psi(t_0, v_0) = 0$ with $v_0 > -1$.
c) $\mathcal{S}$ is closed :
Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathcal{S}$ which is going to $t \in [0,1]$. Set $v_i = v_{t_i}$; by Lemma 1, the v_i are uniformly bounded in $\Lambda_{k+2,\alpha}^s$. Now we use the compactness of the injection $\Lambda_{k+2,\alpha}^s \longrightarrow \Lambda_{k+2,\beta}^t$ when $t \in]0,s[$ and $\beta \in]0,\alpha[$ to extract a subsequence which tends in $\Lambda_{k+2,\beta}^t$ toward $v \in \Lambda_{k+2,\alpha}^s$ with $v > -1$. This v is solution of (E_t).

3 Dimension 2

The results are similar. In fact, we work with the equation

$$(\triangle + A)v = \mathrm{e}^v(f - A) + A(1+v),$$

because when a conformal metric is written in the form $H = \mathrm{e}^{2g}H_0$, we have

$$Scal(H) = \mathrm{e}^{-2g}(S_0 + 2\triangle g).$$

4 An obstruction

The condition $K > s(s-(n-1))$ can be written $s_1 < s < s_2$. Now if $s = s_2 + q$ with $q \in \mathbb{N}\backslash\{0\}$, the solution of

$$(\triangle + K)\varphi = \rho^s$$

who cancels on ∂B is of the form

$$\varphi = \rho^{s_2} \sum_{j=0}^{q-1} a_j \rho^j$$

with $a_j > 0$. So it isn't in Λ^k for all $k > s_2$.

Proposition 4 *We suppose $n \geq 6$, $s > n+1$ and $\exists \epsilon > 0$ so that $\sigma \geq \epsilon \rho^s$. Then if v is a solution of the equation $Scal[(1+v)^{\frac{4}{n-2}} H_0] = S_0 + \sigma$ with $v = 0$ on ∂B and $-1 < v \leq 1$, v cannot be in Λ_0^k if $k > n+1$.*

Proof. By the maximum principle, we have $v \geq 0$. Now the equation is

$$\triangle v + [n + \frac{n}{2}(\frac{n+2}{n-2})v]v = \frac{n-2}{4(n-1)}\sigma(1+v)^p + G(v),$$

with $G(v) \geq 0$ because $p := \frac{n+2}{n-2} \in [1,2]$ (and $v \geq 0$). As $n \geq 4$ and $v \leq 1$, the left term is less than $(\triangle + 2n + 2)v$, and we conclude by the maximum principle.

This example gives us an obstruction for the scalar conformal curvature equation, which is compatible with [11].

For other recent results on the conformal scalar curvature, one can see [2], [4], [8], [9], [13].

References

[1] Th. Aubin, *Nonlinear analysis on manifold. Monge-Ampère Equations*, Springer-Verlag (1982).

[2] P. Aviles, R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Diff. Geom. 21 (1985), 269-281.

[3] A. Besse, *Einstein manifolds*, Springer-Verlag (1987).

[4] C.Y. Chen , K.S. Chen, W.N. Yu, *Conformal Deformations of Metrics on $H^n(-1)$ with Prescribed Scalar Curvature*, Chinese J. Math. 16, $N^0 3$ (1988), 157-187.

[5] E. Delay, *Analyse précisée d'équations semi-linéaires elliptiques sur l'espace hyperbolique et application à la courbure scalaire conforme*, Bull. Soc. Math. France. 125 (1997), 345-381.

[6] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order, 2^{nd} edition*, Springer-Verlag (1983).

[7] C.R. Graham, J.M. Lee, *Einstein metrics with prescribed infinity on the ball*, Adv. in Math. 87, $N^0 2$ (1991), 186-225.

[8] M.C. Leung, *Pinching theorem on asymptotically hyperbolic spaces*, Internat. J. Math. 4, $N^0 5$ (1993), 841-857.

[9] P. Li, L.F. Tam, D.G. Yang, *On the elliptic equations* $\triangle u + ku - Ku^p = 0$ *on complete Riemannian manifolds and their geometric applications* (preprint 1994).

[10] M. Min-oo, *Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*, Math.Ann. 285 (1989), 527-539.

[11] M. Min-oo, Erratum 1997 (en préparation).

[12] W.-M. Ni, *On the elliptic equation* $\triangle u + K(x)u^{\frac{n+2}{n-2}} = 0$*, its generalisation, and application in geometry*, Indiana. Univ. Math. J. 31, $N^0 4$ (1982), 493-529.

[13] A. Ratto, M. Rigoli, L. Vernon, *Scalar curvature and conformal deformations of noncompact Riemannian manifolds* (preprint 1995).

Université de Nice-Sophia Antipolis
Laboratoire J. A. Dieudonné
Mathématiques, Parc Valrose
06108 NICE CEDEX 2
e-mail : <delay@math.unice.fr>