Another Proof of the Krishnaprasad-Berenstein's Theorem

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Abstract

In a very nice paper Krishnaprasad and Berenstein[1] have proved using a geometric argument namely Lasalle's theorem that the equilibrium state $(0,0,0,0)$ of the rigid body dynamics with a single rotor is nonlinear stable. The goal of my paper is to give another proof of this result via the energy-Casimir method.

1 Introduction

Let us consider a rigid body carying a single rotor about the third principal axis. We adopt here the following notation. The rigid body has the moments of inertia $I_1 > I_2 > I_3$, as usual, and suppose that the symmetric rotor is aligned with the principal axis and has the moments of inertia $J_1 = J_2$ and J_3. Let ω_i, $i = 1, 2, 3$ be the carrier body angular velocities and let $\dot{\alpha}$ denote the velocity of the vector(relative to a frame fixed of the center body). Let

$$diag(\lambda_1, \lambda_2, \lambda_3) = diag(J_1 + I_1, J_2 + I_2, J_3 + I_3)$$

be the locked inertia tensor. Then the equations of motion with internal torque (control) can be written as:

$$\begin{cases} \dot{m}_1 = -(m_3 + l)\frac{m_2}{\lambda_2} + \frac{m_2 m_3}{\lambda_3} \ , \\ \dot{m}_2 = (m_3 + l)\frac{m_1}{\lambda_1} + \frac{m_1 m_3}{\lambda_3} \ , \\ \dot{m}_3 = \frac{m_1 m_2}{\lambda_2} + \frac{m_1 m_2}{\lambda_1} \ , \\ \dot{l} = 0 \ . \end{cases}$$

Let $\hat{e} = \{\overline{M_1, M_2, M_3, L,}\}$ denote the standard basis for the known Lie algebra $G = so(3) \times \mathbb{R} \simeq \mathbb{R}^3 \times \mathbb{R} \simeq \mathbb{R}^4$, obeying the commutation rules:

$$\left[\overline{M}_1, \overline{M}_2\right] = \overline{M}_3 \ ,$$

$$\left[\overline{M}_2, \overline{M}_3\right] = \overline{M}_1 \ ,$$

$$\left[\overline{M}_3, \overline{M}_1\right] = \overline{M}_2 \ ,$$

$$\left[\overline{M}_i, L\right] = 0, \forall i = 1, 2, 3 \ .$$

Define a new basis $\hat{e}_2 = \{M_1, M_2, M_3, L\}$, by setting:

$$M_1 = \overline{M}_1 \ ,$$

$$M_2 = \overline{M}_2 \ ,$$
$$M_3 = \overline{M}_3 - L \ ,$$
$$L = \overline{L} \ .$$

With respect to the new basis $\hat{e}_2$, the commutation rules become:

$$[M_1, M_2] = \left[\overline{M}_1, \overline{M}_2\right] = \overline{M}_3 = M_3 + L \ ,$$
$$[M_2, M_3] = \left[\overline{M}_2, \overline{M}_3 - \overline{L}\right] = \overline{M}_1 = M_1 \ ,$$
$$[M_3, M_1] = \left[\overline{M}_3, \overline{M}_1\right] = \overline{M}_2 = M_2 \ ,$$
$$\left[\overline{M}_i, L\right] = 0, \forall i = 1, 2, 3 \ .$$

Denote $\hat{e} = \{X_1, X_2, X_3, X_4\}$, where:

$$X_1 = M_1 \ , \ X_2 = M_2 \ , \ X_3 = M_3 \ , \ X_4 = L \ .$$

Then the structure constants for the Lie algebra G^* in the $\hat{e}$ basis are:

$$[X_1, X_2] = c_{12}^k X_k = c_{12}^1 X_1 + c_{12}^2 X_2 + c_{12}^3 X_3 + c_{12}^4 X_4 = X_3 + X_4 \ ,$$
$$[X_1, X_3] = c_{13}^k X_k = c_{13}^1 X_1 + c_{13}^2 X_2 + c_{13}^3 X_3 + c_{13}^4 X_4 = -X_2 \ ,$$
$$[X_1, X_4] = c_{14}^k X_k = c_{14}^1 X_1 + c_{14}^2 X_2 + c_{14}^3 X_3 + c_{14}^4 X_4 = 0 \ ,$$
$$[X_2, X_3] = c_{23}^k X_k = c_{23}^1 X_1 + c_{23}^2 X_2 + c_{23}^3 X_3 + c_{23}^4 X_4 = X_1 \ ,$$
$$[X_2, X_4] = c_{24}^k X_k = c_{24}^1 X_1 + c_{24}^2 X_2 + c_{24}^3 X_3 + c_{24}^4 X_4 = 0 \ ,$$
$$[X_3, X_4] = c_{34}^k X_k = c_{34}^1 X_1 + c_{34}^2 X_2 + c_{34}^3 X_3 + c_{34}^4 X_4 = 0 \ .$$

Thus

$$c_{12}^1 = 0 \ , \ c_{12}^2 = 0 \ , \ c_{12}^3 = 1 \ , \ c_{12}^4 = 1 \ ,$$
$$c_{13}^1 = 0 \ , \ c_{13}^2 = -1 \ , \ c_{13}^3 = 0 \ , \ c_{13}^4 = 0 \ ,$$
$$c_{14}^1 = 0 \ , \ c_{14}^2 = 0 \ , \ c_{14}^3 = 0 \ , \ c_{14}^4 = 0 \ ,$$
$$c_{23}^1 = 1 \ , \ c_{23}^2 = 0 \ , \ c_{23}^3 = 0 \ , \ c_{23}^4 = 0 \ ,$$
$$c_{24}^1 = 0 \ , \ c_{24}^2 = 0 \ , \ c_{24}^3 = 0 \ , \ c_{24}^4 = 0 \ ,$$
$$c_{34}^1 = 0 \ , \ c_{34}^2 = 0 \ , \ c_{34}^3 = 0 \ , \ c_{34}^4 = 0 \ .$$

Now, let $\overset{\wedge}{e^*}$ be the basis of G^* which is dual to $\hat{e}$. Denote the corresponding coordinates for G^* as:

$$\{x_1, x_2, x_3, x_4\} = \{m_1, m_2, m_3, l\} \ .$$

Then, the minus Lie-Poisson structure on G^* is given by:

$$\left[\{x_i, x_j\}_{LP}^{-}\right] = -\left[c_{ij}^k x_k\right] = \begin{bmatrix} 0 & -(m_3 + l) & m_2 & 0 \\ m_3 + l & 0 & -m_1 & 0 \\ -m_2 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Moreover, an Casimir of the $(\mathbb{R}^4, \{\bullet,\bullet\}^-_{LP})$ configuration takes the form:

$$C(m_1,m_2,m_3,l)=\frac{1}{2}\left[m_1^2+m_2^2+(m_3+l)^2\right] \ .$$

It is easy to see that the dynamics of our system is a Hamilton-Poisson dynamics with:

$$P=so(3)^*\times\mathbb{R}\simeq\mathbb{R}^4 \ ,$$

$$\{\bullet,\bullet\}=\{\bullet,\bullet\}^-_{LP} \ ,$$

$$H(m_1,m_2,m_3,l)=\frac{1}{2}\left[\frac{m_1^2}{\lambda_1}+\frac{m_2^2}{\lambda_2}+\frac{m_3^2}{\lambda_3}\right] \ .$$

2 Krishnaprasad-Berenstein's theorem

Now we can prove via the energy-Casimir metod(see for details Marsden and Raţiu[2] or Puta[3])

Theorem 2.1 *The equilibrium state* $(0,0,0,0)$ *is nonlinear stable.*

Proof. Let H_C be the energy-Casimir function given by:

$$H_C(m_1,m_2,m_3,l)=\frac{1}{2}\left[\frac{m_1^2}{\lambda_1}+\frac{m_2^2}{\lambda_2}+\frac{m_3^2}{\lambda_3}\right]+\varphi\left(\frac{1}{2}\left[m_1^2+m_2^2+(m_3+l)^2\right]\right) \ .$$

Then, the first variation takes the form:

$$\delta\left(H_C\right)=\frac{m_1}{\lambda_1}\delta m_1+\frac{m_2}{\lambda_2}\delta m_2+\frac{m_3}{\lambda_3}\delta m_3+$$

$$+\overset{\bullet}{\varphi}\left[m_1\delta m_1+m_2\delta m_2+(m_3+l)\delta m_3+(m_3+l)\delta l\right] \ .$$

Obviously,

$$\delta\left(H_C\right)(0,0,0,0)=0 \ .$$

On the other hand the second variation is given by:

$$\delta^2\left(H_C\right)=\frac{1}{\lambda_1}(\delta m_1)^2+\frac{1}{\lambda_2}(\delta m_2)^2+\frac{1}{\lambda_3}(\delta m_3)^2+$$

$$+\overset{\bullet\bullet}{\varphi}\left[m_1\delta m_1+m_2\delta m_2+(m_3+l)(\delta m_3+\delta l)\right]^2$$

$$+\overset{\bullet}{\varphi}\left[(\delta m_1)^2+(\delta m_2)^2+(\delta m_3+\delta l)^2\right]$$

and if $\overset{\bullet}{\varphi}(0)>0$ we can say that the second variation at the equilibrium of interest, i.e. $\delta^2(H_C)(0,0,0,0)$, is positive definite and therefore the equilibrium state $(0,0,0,0)$ is nonlinear stable.

References

[1] P. S. Krishnaprasad, *Lie-Poisson structures, dual-spin spacecraft and asymptotic stability*, Nonlinear Analysis, Theory, Methods and Applications, vol. 9 (1985).

[2] J. Marsden and T. Raţiu, *Introduction to Mechanics and Symmetry*, Springer, 1994.

[3] M. Puta, *Hamiltonian Mechanics and Geometric Quantization*, Kluwer Academic Publishers, vol. 270, 1993.

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