An Introduction to Integral Geometry in the n-Dimensional Quaternionic Space

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Abstract

In order to complete the study of Integral Geometry in symmetric spaces of rank one, we introduce in this paper some results of integral geometry in the n-dimensional quaternionic space Q^n. In particular, we define the density for linear spaces in Q^n and the kinematic density of the group of quaternionic motions. [4]

1 Introduction

Integral Geometry in Euclidean space has been extensively studied in the bibliography [see for example L. Santaló: *Integral Geometry and Geometric probability*, Addison–Wesley, 1976]. Two of the necessary elements for developping this theory are the measure of linear spaces and the measure of the group of motions. These measures are closely related whit the invariant measures of the orthogonal group $O(n)$ and of the real Grassmannian manifolds. Integral Geometry in spaces of constant curvature (non-Euclidean Integral Geometry) has been considered in [8] too.

On the other hand Naveira [5] extended to C^n several results of Integral Geometry in R^n, while Ishihara [3] and Gual and Naveira [6] have found interesting results in the n-dimensional complex projective space. Therefore, invariant measures for complex spaces (linear and projective) and for the group of complex motions have been obtained via the invariant measures of the unitary group $U(n)$ and of the complex Grassmannian.

Following with the study of integral geometry in symmetric spaces of rank one, we introduce in this paper some results of integral geometry in the n-dimensional quaternionic space Q^n. In particular, we define the density for linear spaces in Q^n and the kinematic density of the group of quaternionic motions. These densities will be obtained from the invariant measures of the symplectic group $Sp(n)$ and of the quaternionic Grassmannian.

2 Preliminaries

We will denote Q the algebra of quaternions [2], that is the 4-dimensional real algebra with a basis $\{1, i, j, k\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ik = -ki = j$ and $jk = -kj = i$.

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Therefore a quaternion $q \in Q$ may be expressed as $q = a + ib + jc + kd$, with $a, b, c, d \in \mathbb{R}$. Addition and multiplication are defined by the usual distributivity laws.

If we call $c(q) = a - ib - jc - kd$ the conjugate and $N(q) = (qc(q))^{1/2} = (a^2 + b^2 + c^2 + d^2)^{1/2}$ the norm of q, then every non null quaternion q has an inverse given by $c(q)/N(q)$.

Any quaternion $q = a+ib+jc+kd$ can be identified like a pair of complex numbers, namely $q = \alpha + j\beta$ with $\alpha = a + ib$ and $\beta = c - id$. In this way we obtain $zj = j\bar{z}$ for $z \in \mathbb{C}$, this shows us the non-commutativity of Q, and $c(q) = \bar{\alpha} - j\beta$.

Let Q^n be the cartesian product $Q \times \cdots \times Q$. An element $(p^1, \cdots, p^n) \in Q^n$ will be called a quaternionic vector with coordinates $p^1, \cdots, p^n$. The interior simplectic product on Q^n is given by [7]

$$((p^1, \cdots, p^n), (q^1, \cdots, q^n)) = c(p^1)q^1 + \cdots + c(p^n)q^n;$$

also, it is possible to define on Q^n the symmetric interior product [4]

$$< P, Q >= \frac{1}{2}\left((P, Q) + (Q, P)\right).$$

Let $Sp(n)$ be the compact Lie group of linear automorphisms of Q^n that preserve the interior symplectic product on Q^n [7]. It can be identified with the subgroup of $GL(2n, \mathbb{C})$ of matrices

$$\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix},$$

with $A, B \in GL(n, \mathbb{C})$ such that $c(A + jB)^t(A + jB) = I$.

Taking into account that a matrix $C + iD \in GL(n, \mathbb{C})$ can be represented as a matrix

$$\begin{pmatrix} C & D \\ -D & C \end{pmatrix}$$

in $GL(2n, \mathbb{R})$, the Lie group $Sp(n)$ can be considered like a subgroup of $GL(4n, \mathbb{R})$.

3 A density for the group of quaternionic motions

The study of a density for the group $M = Sp(n) \times Q^n$ of quaternionic motions can be done by the method of moving frames [8, p. 180].

If we consider Q^n like the $4n$-dimensional real vector space with the previous symmetric interior product, then a moving frame in an open set $U \subset Q^n$ consists of $4n$ Q^n-valued smooth functions

$$e_a, ie_a, je_a, ke_a, \quad a = 1, \ldots, n$$

on U such that $(e_a(p), ie_a(p), je_a(p), ke_a(p))$ is a unitary frame for all $p \in U$. This moving frame will be denoted $(p, e_a, e^i_a, e^j_a, e^k_a)$.

If we consider an orthonormal fixed frame, to each motion g in Q^n corresponds a moving frame $(p, e_a, e^i_a, e^j_a, e^k_a)$ and reciprocally.

Let us define the following 1-forms for $r, s \in \{i, j, k\}$:

$$\begin{array}{lll}\omega_a =< dp, e_a >, & \omega_a^r =< dp, e_a^r >, & \\ \omega_{ab} =< de_a, e_b >, & \omega_{ab}^{\ r} =< de_a, e_b^r >, & \omega_{ab}^{rs} =< de_a^r, e_b^s > .\end{array}$$

It is easy to prove that

$$\omega_a, \omega_a^i, \omega_a^j, \omega_a^k, \omega_{aa}^{\ i}, \omega_{aa}^{\ j}, \omega_{aa}^{\ k}, \omega_{ab}, \omega_{ab}^{\ i}, \omega_{ab}^{\ j}, \omega_{ab}^{\ k}$$

is a system of Mauren-Cartan forms on Q^n. Then, the density for the group Q^n of quaternionic traslations is

$$dT(n) = \bigwedge_a \omega_a \wedge \omega_a^i \wedge \omega_a^j \wedge \omega_a^k \ ,$$

while the density for the group $Sp(n)$ of quaternionic rotations is

$$dR(n) = \bigwedge_{a<b} \omega_{aa}^{\ i} \wedge \omega_{aa}^{\ j} \wedge \omega_{aa}^{\ k} \wedge \omega_{ab} \wedge \omega_{ab}^{\ i} \wedge \omega_{ab}^{\ j} \wedge \omega_{ab}^{\ k}.$$

Therefore, the density for the group of quaternionic motions is $dK = dT(n) \wedge dR(n)$. We call dK the kinematic density.

From the real representations of C^n and Q^n, the volumes of the groups which preserve the respective interior products in R^n, C^n and Q^n are related as follows. For the Euclidean space we have that this volume is [8]

$$V(O(n)) = \prod_{i=1}^{n} V(S^{i-1});$$

for the complex case we have

$$V(U(n)) = \prod_{i=1}^{n} V(S^{2i-1}).$$

Finally, for the quaternionic space we know from [7] that $Sp(n)/Sp(n-1) = S^{4n-1}$, then [1] $V(Sp(n)) = V(Sp(n-1)).V(S^{4n-1})$. Taking into account that $Sp(1) = S^3$ we have

$$V(Sp(n)) = \prod_{i=1}^{n} V(S^{4i-1}).$$

4 Densities for linear subspaces in Q^n

We want to define a density for the set of r-planes in Q^n, invariant under the group of motions.

Let L_r be a r-dimensional plane through the point $p \in Q^n$. We denote h_r^Q the closed subgroup of motions in Q^n that leave invariant a fixed subspace L_r^0. Since there exists a bijective correspondence between the set of r-planes and the elements of the homogeneous space M/h_r^Q, the problem of finding an invariant density for sets of r-planes is equivalent to that of finding an invariant density on M/h_r^Q.

Proposition 1 *The density for r-planes is given by*

$$dL_r = \bigwedge_{a,\alpha} \omega_{a\alpha} \wedge \omega_{a\alpha}^{\ i} \wedge \omega_{a\alpha}^{\ j} \wedge \omega_{a\alpha}^{\ k} \wedge \omega_\alpha \wedge \omega_\alpha^i \wedge \omega_\alpha^j \wedge \omega_\alpha^k \ ,$$

with $a = 1, \ldots, r$, $\alpha = r+1, \ldots, n$.

Proof. Assume that L_r^0 is the r-plane spanned by the point p and the $4r$ unit vectors e_a, e_a^i, e_a^j, e_a^k, with $a = 1, \ldots, r$. If L_r^0 is kept fixed, then the point and the vectors can vary on L_r^0 and therefore we have

$$< dp, e_\alpha >=< dp, e_\alpha^i >=< dp, e_\alpha^j >=< dp, e_\alpha^k >= 0$$

and in an analogous way

$$< de_a, e_\alpha >=< de_a, e_\alpha^i >=< de_a, e_\alpha^j >=< de_a, e_\alpha^k >= 0.$$

We apply now general methods for getting densities in homogeneus spaces (see [8, p. 166]) and the proof is a consequence of the relations among i, j, k and the fact that $d(dL_r) = 0$. $\square$

In a similar way we have

Proposition 2 *A density for the set of the r-planes that contain a fixed q-plane* ($r \geq q$) *passing through the origin is given by*

$$dL_r[q] = \bigwedge_{a,\alpha} \omega_{a\alpha} \wedge \omega_{a\alpha}^{\ i} \wedge \omega_{a\alpha}^{\ j} \wedge \omega_{a\alpha}^{\ k} \ ,$$

with $a = q+1, \ldots, r$, $\alpha = r+1, \ldots, n$.

For $q = 0$ we get the density for r-planes about a fixed point, that is

$$dL_r[0] = \bigwedge_{a,\alpha} \omega_{a\alpha} \wedge \omega_{a\alpha}^{\ i} \wedge \omega_{a\alpha}^{\ j} \wedge \omega_{a\alpha}^{\ k}, \quad a = 1, \ldots, r, \alpha = r+1, \ldots, n.$$

If we consider the quaternionic Grassmann manifold $G_{r,n-r}$, that is the manifold of the r-planes of Q^n through a fixed point, we obtain its volume element in the following result.

Corollary 1

$$dG_{r,n-r} = \bigwedge_{a,\alpha} \omega_{a\alpha} \wedge \omega_{a\alpha}^{\ i} \wedge \omega_{a\alpha}^{\ j} \wedge \omega_{a\alpha}^{\ k}, \quad a = q+1, \ldots, r, \ \alpha = r+1, \ldots, n.$$

Proposition 1 gives an expresion for dL_r. It is useful to introduce the elements of the $(n-r)$-plane perpendicular to L_r through a fixed point 0 in the expression for dL_r.

Let $L_{n-r}[0]$ be the $n-r$ plane perpendicular to L_r through 0 and $P = L_r \cap L_{n-r}[0]$. Choose the orthonormal frame

$$(e_a, e_a^i, e_a^j, e_a^k, e_\alpha, e_\alpha^i, e_\alpha^j, e_\alpha^k), \quad a = 1, \ldots, r, \ \alpha = r+1, \ldots, n$$

such that e_a, e_a^i, e_a^j, e_a^k are in L_r and $e_\alpha, e_\alpha^i, e_\alpha^j, e_\alpha^k$ are in the orthogonal plane to L_r. One has that

$$\omega_\alpha \wedge \omega_{a\alpha}^{\ i} \wedge \omega_{a\alpha}^{\ j} \wedge \omega_{a\alpha}^{\ k} =< dP, e_\alpha > \wedge < dP, e_\alpha^i > \wedge < dP, e_\alpha^j > \wedge < dP, e_\alpha^k >$$

is the volume element of $L_{n-r}[0]$ at P. It will be denoted by $d\sigma_{n-r}$. Taking into account Proposition 1 and Proposition 2 we have

Proposition 3 *The density for the r-planes is given by*

$$dL_r = d\sigma_{n-r} \wedge dL_{n-r}[0] = d\sigma_{n-r} \wedge dL_r[0] \ .$$

Example. If $r = 1$ in Proposition 3, then $dL_1 = d\sigma_{n-1} \wedge dL_{n-1}[0]$. We have that $dL_{n-1}[0]$ is the density dHP of the projective quaternionic space.

Example. If $r = n - 1$ in Proposition 3, then

$$dL_{n-1} = d\sigma_1 \wedge dL_1[0] = d\sigma_1 \wedge dL_{n-1}[0] = d\sigma_1 \wedge dHP.$$

Proposition 4 *Let us to consider sets of pairs of linear spaces* $(L_r, {L_a}^r)$ *such that* ${L_a}^r \subset L_r$, $(a \leq r)$. *Then*

$$d{L_a}^{(r)} \wedge dL_r = dL_r[a] \wedge dL_a \ .$$

Proof.

Choose the moving frame $(p, e_1, \ldots, e_n, e_1^i, \ldots, e_n^i, e_1^j, \ldots, e_n^j, e_1^k, \ldots, e_n^k)$ so that

$$p, e_1, \ldots, e_a, e_1^i, \ldots, e_a^i, e_1^j, \ldots, e_a^j, e_1^k, \ldots, e_a^k$$

belong to ${L_a}^{(r)}$ and

$$p, e_1, \ldots, e_r, e_1^i, \ldots, e_r^i, e_1^j, \ldots, e_r^j, e_1^k, \ldots, e_r^k$$

belong to L_r. Then

$$dL_r = \bigwedge_{b,h,\alpha} \omega_{b\alpha} \wedge \omega_{b\alpha}^{\ i} \wedge \omega_{b\alpha}^{\ j} \wedge \omega_{b\alpha}^{\ k} \wedge \omega_\alpha \wedge \omega_\alpha^i \wedge \omega_\alpha^j \wedge \omega_\alpha^k \wedge \omega_{h\alpha} \wedge \omega_{h\alpha}^{\ i} \wedge \omega_{h\alpha}^{\ j} \wedge \omega_{h\alpha}^{\ k},$$

$$d{L_a}^{(r)} = \bigwedge_{b,h} \omega_{bh} \wedge \omega_{bh}^{\ i} \wedge \omega_{bh}^{\ j} \wedge \omega_{bh}^{\ k} \wedge \omega_h \wedge \omega_h^i \wedge \omega_h^j \wedge \omega_h^k,$$

$$dL_r[a] = \bigwedge_{h,\alpha} \omega_{h\alpha} \wedge \omega_{h\alpha}^{\ i} \wedge \omega_{h\alpha}^{\ j} \wedge \omega_{h\alpha}^{\ k},$$

and

$$dL_a = \bigwedge_{b,h,\alpha} \omega_{b\alpha} \wedge \omega_{b\alpha}^{\ i} \wedge \omega_{b\alpha}^{\ j} \wedge \omega_{b\alpha}^{\ k} \wedge \omega_{bh} \wedge \omega_{bh}^{\ i} \wedge \omega_{bh}^{\ j} \wedge \omega_{bh}^{\ k} \wedge \omega_\alpha \wedge \omega_\alpha^i \wedge \omega_\alpha^j \wedge \omega_\alpha^k \wedge \omega_h \wedge \omega_h^i \wedge \omega_h^j \wedge \omega_h^k,$$

with $b = 1, \ldots, i$, $h = a + 1, \ldots, r$, $\alpha = r + 1, \ldots, n$.

Then the result follows immediately. $\square$

5 The kinematic density in Q^n

We have computed before the density dK for the group of quaternionic motions. This section is concerned with this kinematic density and we generalize a useful expression for dK given in [8] for the real kinematic density.

Let L_h be an h-plane in Q^n. If we denote dK^h the kinematic density in L_h and $dK^{n-h}[x]$ the kinematic density about a point x in the $(n-h)$–plane orthogonal to L_h we have the following expression for dK.

Proposition 5 *The kinematic density can be expressed by*

$$dK = dL_h \wedge dK^h \wedge dK^{n-h}[x] \ .$$

Proof. We have

$$dL_h = \bigwedge_{a,\alpha} \omega_\alpha \wedge \omega_\alpha^i \wedge \omega_\alpha^j \wedge \omega_\alpha^k \wedge \omega_{a\alpha} \wedge \omega_{a\alpha}{}^i \wedge \omega_{a\alpha}{}^j \wedge \omega_{a\alpha}{}^k,$$

with $a = 1, \ldots, h$, $\alpha = h+1, \ldots, n$,

$$dK^h = \bigwedge_{a,b} \omega_a \wedge \omega_a^i \wedge \omega_a^j \wedge \omega_a^k \wedge \omega_{ab} \wedge \omega_{ab}{}^i \wedge \omega_{ab}{}^j \wedge \omega_{ab}{}^k,$$

with $1 \leq a < b \leq h$, and

$$dK^{n-h}[x] = \bigwedge_{\alpha,\beta} \omega_{\alpha\beta} \wedge \omega_{\alpha\beta}{}^i \wedge \omega_{\alpha\beta}{}^j \wedge \omega_{\alpha\beta}{}^k,$$

with $h+1 \leq \alpha < \beta \leq n$. Then the result is obtained by taking the general expression of dK. $\square$

In an analogous way we have

Proposition 6 $dK = dL_h[0] \wedge dK^h \wedge dK^{n-h}$.

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