Notes on Semi-Symmetric Connections on Spaces Endowed with Weyl f-Structures

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Abstract

This paper deals with manifolds endowed with Weyl f-structures. We present an invariant of the transformations of the semi-symmetric connections, compatible with Weyl f-structures and properties of the deformation algebra of some special connections. In the third section we study flat spaces endowed with Weyl f- structures.

1 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold, $G = \{e^{\lambda}g \mid \lambda \in \mathcal{F}(M)\}$ be the conformally structure generated by g and $W : G \mapsto \Lambda^1(M)$ be the Weyl structure, where $W(e^{\lambda}g) = W(g) - d\lambda$, for any $\lambda \in \mathcal{F}(M)$, $\Lambda^1(M)$ being the space of 1-forms on M. (M, G, W) is a Weyl manifold. These spaces introduced by Folland ([2]) were studied by many authors and the topic is still actual ([1], [5]).

Weyl generalized manifolds are natural extensions of these spaces. For a nowhere vanishing mapping $f \in \mathcal{F}(M)$, we define the Weyl f-structure ([8]) $W^f : G \mapsto \Lambda^1(M)$, where $W^f(e^{\lambda}g) = W^f(g) - f d\lambda$, for any $\lambda \in \mathcal{F}(M)$. $(M,\ G,\ W^f)$ is called a space endowed with the Weyl f-structure W^f or a Weyl generalized manifold.

A linear connection ∇ on M is said to be compatible with the Weyl f-structure W^f and associated to the 1-form ω if

$$(\nabla_{fX} g)(Y, Z) + W^f(g)(X)g(Y, Z) + \omega(Y)g(X, Z) + \omega(Z)g(X, Y) = 0\,, \tag{1.1}$$

for any $X\,, Y\,, Z \in \mathcal{X}(M)$

For $f = 1$ and $\omega = 0$ there exists a unique symmetric connection compatible with the Weyl structure W, denoted $\overset{W}{\nabla}$.

Theorem 1.1 (Higa) *The following assertions are equivalent :*

1. *We have $dW(g) = 0$ and $[W(g)] = 0$ in $H^1(M)$;.*
2. *There exists $h \in G$ such that $\overset{W}{\nabla} h = 0$.*

∇ is said to be a σ-semi-symmetric connection if

$$T(X\,, Y) = \sigma(Y)X - \sigma(X)Y\,, \tag{1.2}$$

where σ is a 1-form and T the torsion tensor field of ∇.

We introduce on M the Obata operators for g by

$$\Omega = \frac{1}{2}(I \otimes I - g \otimes \widetilde{g})\,,\ \Omega^* = \frac{1}{2}(I \otimes I + g \otimes \widetilde{g})\,, \tag{1.3}$$

where $\widetilde{g}$ is the (2, 0)-tensor field such that $g \cdot \widetilde{g} = I$, I being the identity (1, 1)-tensor field, $(g \cdot \widetilde{g})(X, \sigma) = g(X, \sigma^{\#})$ $g(X, \sigma^{\#}) = \sigma(X)$, for any $\sigma \in \Lambda^1(M), X \in \mathcal{X}(M)$.

These operators satisfy the identities

$$\Omega + \Omega^* = I \otimes I\,,\ \Omega\Omega = \Omega\,,\ \Omega^*\Omega^* = \Omega^*\,,\ \Omega\Omega^* = \Omega^*\Omega = 0.$$

If a manifold M is endowed with two linear connections ∇ and $\widetilde{\nabla}$, we define $X \star Y = A(X, Y)$, where $A = \nabla - \widetilde{\nabla} \in \mathcal{T}_2^1(M)$. The module $\mathcal{X}(M)$ becomes an algebra, denoted $\mathcal{U}(M, \widetilde{\nabla}, \nabla)$. An element X of this deformation algebra is called characteristic vector field if there exists $\lambda \in \mathcal{F}(M)$ such that $A(X, X) = \lambda X$. If moreover, $X_p \neq 0, \forall p \in M$, X is called a characteristic direction vector field.

2 Transformations of connections

Theorem 2.1 *Let $(M,\ G,\ W^f)$ be a Weyl generalized manifold. There is a unique σ-semi-symmetric connection satisfying* (1.1)*, where $g, f, W^f(g), \omega$ are given, determined by*

$$\begin{aligned} \nabla_{fX}Y =& \overset{\circ}{\nabla}_{fX}\, Y + \frac{1}{2}W^f(g)(X)Y + \left(\frac{1}{2}W^f(g) + f\sigma\right)(Y)X - \\ & -g(X,Y)\left(\frac{1}{2}W^f(g) + f\sigma - \omega\right)^{\#}, \end{aligned} \tag{2.1}$$

$\overset{\circ}{\nabla}$ being the Levi-Civita connection for g.

Proof. Let ∇ be a connection satisfying (1.1) and $\overset{\circ}{\nabla}$ be an arbitrary linear connection. We have $\nabla = \overset{\circ}{\nabla} + A$, where $A \in \mathcal{T}_2^1(M)$. We get

$$\begin{aligned} (\overset{\circ}{\nabla}_{fX}\, g)(Y, Z) - fg(A(X, Y), Z) - fg(A(X, Z), Y) + W^f(g)(X)g(Y, Z) + \\ +\omega(Y)g(X, Z) + \omega(Z)g(X, Y) = 0\,. \end{aligned}$$

The previous relation is equivalent with

$$f(\Omega^*A)(X, Y) = \frac{1}{2}\left[(\overset{\circ}{\nabla}_{fX}\, g)^{\#}_{Y} + W^f(g)(X)Y + g(X, Y)\omega^{\#} + \omega(Y)X\right].$$

This equation has the solution

$$\begin{aligned} A(X, Y) = \frac{1}{2}\left[(\overset{\circ}{\nabla}_{X}\, g)^{\#}_{Y} + \frac{1}{f}W^f(g)(X)Y + \frac{1}{f}g(X, Y)\omega^{\#} + \frac{1}{f}\omega(Y)X\right] + \\ +(\Omega U)(X, Y)\,, \end{aligned}$$

where U is an arbitrary (1,2)-tensor field. If $\overset{\circ}{\nabla}$ is the Levi-Civita connection of g, replacing $U(X, Y) = \frac{1}{f}W^f(g)(Y)X + V(X, Y)$, for an arbitrary $(1, 2)$-tensor field V, we get

$$\nabla_{fX}Y = \overset{\circ}{\nabla}_{fX}\, Y + \frac{1}{2}W^f(g)(X)Y + \left(\frac{1}{2}\omega + \frac{1}{2}W^f(g)\right)(Y)X -$$

$$-g(X,Y)\left(\frac{1}{2}W^f-\frac{1}{2}\omega\right)^{\#}+(\Omega V)(X,Y)\,. \tag{2.2}$$

From (2.2) we get $fT(X,Y)=(\Omega V)(X,Y)-(\Omega V)(Y,X)+\frac{1}{2}\omega(Y)X-\frac{1}{2}\omega(X)Y$ and using (1.2) we have

$$(\Omega V)(X,Y)=\left(f\sigma-\frac{1}{2}\omega\right)(Y)X-g(X,Y)\left(f\sigma-\frac{1}{2}\omega\right)^{\#}. \tag{2.3}$$

From (2.2) and (2.3) we obtain our assertion.

Remark. If the 1-form σ is arbitrary, then the relation (2.1) determines the set of all linear connections satisfying (1.1), for given $g, f, W^f(g)$.

Theorem 2.2 *The set of all semi-symmetric connections, compatible with Weyl f-structures and associated to some 1-forms, endowed with the transformations of connections is a group $(\mathcal{C}_s,\cdot)$ which is isomorphic with the additive group $(\Lambda^1(M)\otimes\Lambda^1(M)\otimes\Lambda^1(M)\,,+)$.*

Proof. Any $\overline{\sigma}$-semi-symmetric connection $\overline{\nabla}$, compatible with the Weyl $\overline{f}$-structure $\overline{W}^{\overline{f}}$ and associated to the 1-form $\overline{\omega}$, is obtained from a σ-semi-symmetric connection ∇, compatible with the Weyl f-structure W^f and associated to the 1-form ω by the transformation

$$\overline{\nabla}_{\overline{f}X}Y=\nabla_{fX}Y+p(X)Y+q(Y)X-g(X,Y)r^{\#}\,, \tag{2.4}$$

where $p=\frac{1}{2}\left(\overline{W}^{\overline{f}}(g)-W^f(g)\right), q=p+\overline{f}\overline{\sigma}-f\sigma, r=q-\overline{\omega}+\omega$ and $g\in G$ is fixed. If we consider $\nabla\mapsto\overline{\nabla}$ and $\overline{\nabla}\to\overline{\overline{\nabla}}$, then the product of transformations $\nabla\mapsto\overline{\overline{\nabla}}$ belongs to $\mathcal{C}_s$. We have the property of associativity. The null element is given by (2.4) for $p=r=q=0$ and the inverse for $\nabla\mapsto\overline{\nabla}$ is the transformation given by (2.4) for $(-p,-q,-r)$. $(\mathcal{C}_s\,,\ \cdot)$ is a subgroup of the abelian group $(\mathcal{C}\,,\ \cdot)$, $\mathcal{C}$ being the set of all linear connections compatible with Weyl f-structures and associated to some 1-forms.

So $\Psi:\Lambda^1(M)\otimes\Lambda^1(M)\otimes\Lambda^1(M)\to\mathcal{C}_s,\ \Psi(p,q,r)=(\nabla\mapsto\overline{\nabla})$, given by (2.4) is an isomorphism.

Theorem 2.3 *For $n>2$, the tensor field*

$$C^i_{jkl}=R^{*\,i}_{\;jkl}+\frac{2}{n-2}\left\{\Omega^{mi}_{jk}\left(R^*_{\;ml}-\frac{K}{2(n-1)}g_{ml}\right)-\right.$$

$$\left.-\Omega^{mi}_{jl}\left(R^*_{\;mk}-\frac{K}{2(n-1)}g_{mk}\right)\right\} \tag{2.5}$$

where K is the scalar curvature, $R^{*i}_{jkl} = R^i_{jkl} - \frac{1}{n}\delta^i_j B_{hk}$, $B_{hk} = R^s_{shk}$ and $R^*_{jk} = R^{*i}_{jki}$, is an invariant to the transformation of a σ-semi-symmetric connection, compatible with a Weyl f-structure W^f to a $\overline{\sigma}$-semi-symmetric connection $\overline{\nabla}$, compatible with a Weyl f-structure $\overline{W}^f$.

Proof. From (2.4) we have $f\overline{\Gamma}^h_{jk} = f\Gamma^h_{jk} + \delta^h_j p_k + \delta^h_k q_j - g_{jk}g^{hm}q_m$ and $\overline{R}^i_{jkl} = R^i_{jkl} + \delta^i_j(p'_{k/l} - p'_{l/k}) + \delta^i_j T^m_{kl}p_m + 2\Omega^{mi}_{jk}\left(q'_{m/l} - q'_m q'_l + \frac{1}{2}g_{ml}\rho\right) - 2\Omega^{mi}_{jl}\left(q'_{m/k} - q'_m q'_k + \frac{1}{2}g_{mk}\rho\right) + 2\Omega^{mi}_{js}T^s_{kl}q'_m$, where $\rho = g^{ij}q'_i q'_j$, $fg'_i = q_i$, $fp'_i = p_i$.

Considering $p_{kl} = p'_{k/l} + p'_k\sigma_l$ and $q_{kl} = q'_{k/l} - q'_k q'_l + \frac{1}{2}g_{kl}\rho + q'_k\sigma_l$, ∇ being a σ-semi-symmetric connection, we find

$$\overline{R}^{\,i}_{jkl} = R^{\,i}_{jkl} + \delta^i_j(p_{kl} - p_{lk}) + 2\Omega^{mi}_{jk}q_{ml} - 2\Omega^{mi}_{jl}q_{mk}\,. \tag{2.6}$$

Using $\Omega^{mi}_{ik} = 0$, we get $\overline{B}_{kl} = B_{kl} + n(p_{kl} - p_{lk})$ and substituting in (2.6) we get

$$\overline{R}^{*\,i}_{jkl} = R^{*\,i}_{jkl} + 2\Omega^{mi}_{jk}q_{ml} - 2\Omega^{mi}_{jl}q_{mk}\,. \tag{2.7}$$

Making the contraction, we have:

$$\overline{R}^*_{jk} = R^*_{jk} - (n-2)q_{jk} - g_{jk}\widetilde{\rho}\,, \tag{2.8}$$

where $\widetilde{\rho} = g^{ij}q_{ij}$. Hence, we get $\widetilde{\rho} = -\dfrac{\overline{K} - K}{2(n-1)}$. Substituting in (2.8), we obtain

$$q_{jk} = -\frac{1}{n-2}\left\{\overline{R}^*_{jk} - R^*_{jk} - g_{jk}\frac{\overline{K} - K}{2(n-1)}\right\}\,. \tag{2.9}$$

From (2.7) and (2.8) we find $C^{\,i}_{jkl} = \overline{C}^{\,i}_{jkl}$.

Theorem 2.4 *Let $(M\,, G\,, W^f)$ be a space endowed with a Weyl f-structure and ∇ a σ-semi-symmetric connection, compatible with the Weyl f-structure W^f.*

If $\sigma^\#_p \neq 0, \forall p \in M$, then the characteristic direction vector fields of $\mathcal{U}(M, \widetilde{\nabla}, \nabla)$ are colinear to $\sigma^\#$, where

$$\widetilde{\nabla}_{fX}Y = \overset{\circ}{\nabla}_{fX}\,Y + \frac{1}{2}W^f(g)(X)Y + \frac{1}{2}W^f(g)(Y)X - g(X,Y)\left(\frac{1}{2}W^f(g)\right)^{\#}.$$

Proof. From (2.1) we get $\nabla_X Y - \widetilde{\nabla}_X Y = \sigma(Y)X - g(X,Y)\sigma^\#$ and $A(X,X) = \sigma(X)X - g(X,X)\sigma^\#, \forall X \in \mathcal{X}(M)$.

If X is a characteristic direction vector field then we obtain $\sigma^\# \otimes X - X \otimes \sigma^\# = 0$, by virtue of which there exists $f \in \mathcal{F}(M)$ such that $X = f\sigma^\#$.

3 Flat spaces endowed with Weyl $\boldsymbol{f}$-structures

Theorem 3.1 *Let (M,G,W^f) be a Weyl generalized manifold and ∇ be the symmetric connection, compatible with the Weyl f-structure W^f.*

The following assertions are equivalent:

1. *There exist the local 1-forms p and q such that the space endowed with the semi-symmetric connection*

   (3.1) $$\overline{\nabla}_{fX}Y = \nabla_{fX}Y + q(Y)X + p(X)Y - g(X,Y)q^{\#}$$

   is flat;

2. *The tensor C is zero.*

Proof. If there exist p_k and q_k such that $\overline{R}^{\,i}_{jkl} = 0$, then $\overline{R}^{*i}_{jkl} = 0$ and hence $\overline{C}^{\,i}_{jkl} = C^i_{jkl} = 0$.

Conversely, if $C^i_{jkl} = 0$, we consider the systems of equations

(3.2) $$\begin{cases} p'_{k/l} = p_{kl} \ , \\ p_{kl} - p_{lk} = -\frac{1}{n}B_{kl} \ , \end{cases}$$

(3.3) $$\begin{cases} q'_{k/l} = q_{kl} + q'_k q'_l - \frac{1}{2}g_{kl}\rho \ , \\ q_{kl} = \dfrac{1}{n-2}\left[R^*_{kl} - \frac{K}{2(n-1)}g_{kl}\right] \ , \end{cases}$$

where $\rho = g^{ij}q_iq_j$.

If (p_k, q_k) is a solution of these systems, then we get

$$\overline{R}^{\,i}_{jkl} = R^i_{jkl} - \frac{1}{n}\delta^i_j B_{kl} + \frac{2}{n-2}\left\{\Omega^{mi}_{jk}\left[R^*_{ml} - \frac{K}{2(n-1)}g_{ml}\right] - \right.$$

$$\left. -\Omega^{mi}_{jl}\left[R^*_{mk} - \frac{K}{2(n-1)}g_{mk}\right]\right\} = C^i_{jkl} = 0.$$

We must prove that if $C^i_{jkl} = 0$, $n > 3$, then the systems (3.2), (3.3) have solutions. Because

$$p'_{k/l} - p'_{l/k} = \Phi_{k/l} - \Phi_{l/k} = -\frac{1}{n}B_{kl} \ ,$$

where

$$\Phi_k = -\frac{1}{2f}\left(W^f(g)\right)_k \ ,$$

we get

(3.4) $$p_k = \Phi_k + \frac{\partial h}{\partial x^k} \, ,$$

h being an arbitrary smooth mapping. From (3.3) we get

$$q'_{j/k/l} - q'_{j/l/k} = q_{jk/l} + q'_{j/l}q'_k + q'_j q'_{k/l} - g_{jk}g^{sm}q_{sl}q'_m -$$

$$-\left(q_{jl/k} + q'_{j/k}q'_l + q'_j q'_{l/k} - g_{jl}g^{sm}q'_{s/k}q'_m\right).$$

From the system (3.3), using $C^i_{jkl} = 0$, we obtain

$$-\frac{q'_j}{n}B_{kl} + 2q'_i\left(\Omega^{mi}_{jk}q_{ml} - \Omega^{mi}_{jl}q_{mk}\right) = q_{jk/l} - q_{jl/k} + q'_k q_{jl} - q'_l q_{jk} -$$

$$-(g_{jk}q_{sl} - g_{jl}q_{sk})g^{sm}q'_m + q'_j(q_{kl} - q_{lk}).$$

Because

$$q_{kl} - q_{lk} = \frac{1}{n-2}\left(R^*_{kl} - R^*_{lk}\right) = -\frac{1}{n}B_{kl},$$

we get the integrability condition for the systems (3.2), (3.3)

$$q_{jk/l} - q_{jl/k} = 0. \tag{3.5}$$

The relation (3.5) is a consequence of the formula (3.4). Indeed, $\sum\limits_{k,l,h}^{C} R^{*\,i}_{\ jkl/h} = 0$. From (3.4) we get

$$\Omega^{mi}_{jl}q_{mk/h} - \Omega^{mi}_{jk}q_{ml/h} + \Omega^{mi}_{jh}q_{ml/k} - \Omega^{mi}_{jl}q_{mh/k} +$$

$$+\Omega^{mi}_{jk}q_{mh/l} - \Omega^{mi}_{jh}q_{mk/l} = 0.$$

Hence $(n-3)(q_{jk/l} - q_{jl/k}) = 0$. Because $n > 3$ we have (3.5).

Theorem 3.2 *Let (M, G, W^f) be a Weyl generalized manifold and ∇ be a semi-symmetric connection, compatible with Weyl f-structure W^f.*

The following assertions are equivalent :

1. *There exist the local 1-forms p, q such that the space endowed with the semi-symmetric connection determined by (3.1) is flat*

2. *The tensor field C is zero.*

Proof. If there exist p and q such that $\overline{R} = 0$, then $C = 0$.

Conversely, the σ-semi-symmetric connection ∇, compatible with the Weyl f-structure W^f can be written

$$\nabla_{fX}Y = \overset{\circ}{\nabla}_{fX} Y + \frac{1}{2}W^f(g)(X)Y + \left(\frac{1}{2}W^f(g) + f\sigma\right)(Y)X -$$

$$-g(X,Y)\left(\frac{1}{2}W^f(g) + f\sigma\right)^{\#},$$

$\overset{\circ}{\nabla}$ being the Levi-Civita connection associated to g. We have $C = W = 0$. The previous theorem applied for $\overset{\circ}{\nabla}$ and $\overline{\nabla}$ implies that there exist local the 1-form u and v such that the space endowed with the connection

$$\overline{\nabla}_{fX}Y = \overset{\circ}{\nabla}_{fX} Y + u(Y)X + v(X)Y - g(X,Y)u^{\#}$$

is flat.

Considering $\nabla \mapsto \overset{\circ}{\nabla} \mapsto \overline{\nabla}$ we get

$$\overline{\nabla}_{fX}Y = \nabla_{fX}Y + \left(u - \frac{1}{2}W^f(g) - f\sigma\right)(Y)X + \left(v - \frac{1}{2}W^f(g)\right)(X)Y -$$

$$-g(X,Y)\left(u - \frac{1}{2}W^f(g) - f\sigma\right)^{\#}.$$

Denoting $q = u - \frac{1}{2}W^f(g) - f\sigma$ and $p = v - \frac{1}{2}W^f(g)$, we get our assertion.

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