On Quantum Vector Fields in General Relativistic Quantum Mechanics

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Abstract

We introduce a Lie algebra of distinguished vector fields on a quantum bundle over general relativistic classical spacetime and exhibit a natural isomorphism with the Lie algebra of quantisable functions on the classical phase space. These constructions have been inspired by analogous results in the Galilei framework related to a new quantisation scheme. The present results are expected to lead to an analogous covariant quantisation procedure in the Einstein framework.[2]

Introduction

In the papers [2, 3, 4] the authors have presented a covariant formulation of classical and quantum mechanics on a curved spacetime with absolute time (sometimes referred to as *Galilei* or *Newtonian theory*), in terms of fibre bundles, jet spaces, general connections and cosymplectic structures (the corresponding theory for a spin particle can be found in [1]).

We expect that the above scheme of covariant quantisation in the general relativistic Galilei framework could inspire analogous results in the standard Einstein general relativistic spacetime. Actually, in the papers [6, 7, 8] we have reformulated in a Lorentz manifold M several results of [2, 3, 4]. In particular, we have analysed the contact structure of the phase space and introduced a Lie algebra of functions, which is a candidate for the quantisation procedure.

In this paper we introduce a Lie algebra of distinguished vector fields on a quantum bundle and exhibit a natural isomorphism with the Lie algebra of quantisable functions on the classical phase space. These vector fields are supposed to lead us to a covariant formulation of quantum operators.

All manifolds and mappings are supposed to be smooth.

We denote the sheaf of local smooth function on a manifold M by $\mathcal{F}(M)$, the sheaf of vector fields on M by $\mathcal{T}(M)$ and the sheaf of 1-forms on M by $\mathcal{T}^*(M)$.

To include "physical dimensions" in our geometrical theory, we assume the following fundamental unit half-spaces [2]:

(1) the positive 1-dimensional semi-vector space $\mathbb{T}$ over $I\!\!R^+$ of *time intervals*,

(2) the positive 1-dimensional semi-vector space $\mathbb{L}$ over $I\!\!R^+$ of *lengths*,

(3) the positive 1-dimensional semi-vector space $\mathbb{M}$ over $I\!\!R^+$ of *masses*.

A *time unit of measurement* is defined to be a base of $\mathbb{T}$ or its dual

$$u_0 \in \mathbb{T}, \qquad u^0 \in \mathbb{T}^*.$$

Moreover, we refer to the *light velocity* and the *Planck constant*

$$c \in \mathbb{T}^* \otimes \mathbb{L}, \qquad \hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M},$$

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and consider a classical particle with *mass* and *charge*

$$m \in \mathbb{M}, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}.$$

1. Geometry of phase space

First we briefly recall the basic geometric properties of the classical phase space of scalar particles in the Einstein general relativistic framework. For further details and proofs, see [7, 8].

1.1. **Phase space.** We start with the basic assumptions and definitions.

We assume *spacetime* to be a 4-dimensional oriented and time-oriented manifold M equipped with a scaled Lorentzian metric of signature $(+---)$

$$g : M \to \mathbb{L}^2 \otimes (T^*M \underset{M}{\otimes} T^*M).$$

Local coordinate charts on M will be denoted by (x^λ), $\lambda = 0, 1, 2, 3$.
The coordinate expression of g is then

$$g = g_{\lambda\mu}\, d^\lambda \otimes d^\mu, \qquad g_{\lambda\mu} : M \to \mathbb{L}^2 \otimes I\!R.$$

In what follows we shall refer to local *spacetime* coordinate charts such that the vector ∂_0 is time-like and time oriented and $\partial_1, \partial_2, \partial_3$ are space-like; hence $g_{00} > 0, g_{11}, g_{22}, g_{33} < 0$.

Latin indices $i, j, p, \dots$ will span space-like coordinates, while Greek indices λ, μ, ϕ, $\dots$ will span spacetime coordinates.

A k-jet of 1-dimensional submanifolds of M at $x \in M$ is defined to be an equivalence class of 1-dimensional submanifolds touching each other at x with a contact of order k, [11]. The k-jet of a 1-dimensional submanifold $l \subset M$ at $x \in l$ is denoted by $j_k l(x)$. The set of all k-jets of 1-dimensional submanifolds of M can be equipped, in a natural way, with a smooth structure. The corresponding manifold is denoted by $J_k(M, 1)$ and its projection on M is denoted by $\pi^k_0 : J_k(M, 1) \to M$.

We have a natural fibred isomorphism over M of the first jet bundle with the Grassmannian bundle of dimension 1

$$J(M, 1) \equiv J_1(M, 1) \to \mathrm{Grass}(M, 1) : \phi \mapsto L_\phi,$$

where $\phi \in J(M, 1)$ and $L_\phi \subset T_{\underline{\phi}}M$ is the tangent space at $\underline{\phi} = \pi^1_0(\phi)$ of 1-dimensional submanifolds generating ϕ.

A *motion* in M is defined to be a 1-dimensional time-like submanifold $l \subset M$.

We define the *phase space* for a classical particle to be the subspace

$$UM \equiv U_1 M \subset J(M, 1)$$

of all 1-jets of motions. Hence, $\phi = j_1 l(\underline{\phi}) \in J(M, 1)$ belongs to UM if and only if $L_\phi = T_{\underline{\phi}} l$ lies inside the light cone.

A motion $l \subset M$ can be expressed locally, in terms of a spacetime chart (x^0, x^i), by a formula of the type

$$x^i = l^i(x^0),$$

i.e. , more precisely, $x^i|l = l^i \circ x^0|l$, with $l^i : I\!R \to I\!R$.

Every spacetime chart on M determines naturally a local fibred chart

$$(x^0, x^i; x^i_0)$$

on $U(M)$. We shall always refer to such charts on the phase space.

1.2. **Jet contact structure.** The geometric structure of the phase space and the Lorentz metric allow us to recover the jet contact structure.

We have the following natural *jet contact maps*

$$\text{д} : UM \to \mathbb{T}^* \otimes TM\,, \qquad \tau^\natural := \frac{g^\flat}{c^2} \circ \text{д} : UM \to \mathbb{T} \otimes T^* M\,,$$

with coordinate expressions

$$\text{д} = c\,\alpha\,\text{д}_0 = c\,\alpha\,(\partial_0 + x^i_0 \partial_i)\,, \qquad \tau^\natural \equiv \tau^\natural_\lambda\, d^\lambda = \frac{\alpha}{c}\,(g_{0\lambda} + g_{i\lambda} x^i_0)\, d^\lambda\,,$$

where

$$\alpha = 1/\|\text{д}_0\| = 1/\sqrt{g_{00} + 2 g_{0j} x^j_0 + g_{ij} x^i_0 x^j_0} \in \mathbb{L}^*\,.$$

We have

$$\text{д} \lrcorner\, \tau^\natural = 1\,,$$

i.e. , in coordinates,

$$c\,\alpha\,(\tau^\natural_0 + \tau^\natural_h x^h_0) = 1\,.$$

1.3. **Splitting of the tangent space of spacetime.** The jet contact structure induces a natural splitting, over the phase space, of the tangent space of spacetime into two orthogonal components.

We define the vector bundles over UM

$$T^{\|}M := \{(\phi, X) \in UM \underset{M}{\times} TM \ \mid\ X \in L_\phi\}\,,$$
$$T^{\perp}M := \{(\phi, X) \in UM \underset{M}{\times} TM \ \mid\ X \in L^\perp_\phi\}\,,$$

which yield the splitting

$$UM \underset{M}{\times} TM = T^{\|}M \underset{UM}{\oplus} T^\perp M\,.$$

We observe that

$$T^{\|}M = \{(\phi, X) \in UM \underset{M}{\times} TM \ \mid\ X \in \text{д}(\phi)(\mathbb{T})\}\,,$$
$$T^{\perp}M = \{(\phi, X) \in UM \underset{M}{\times} TM \ \mid\ X \lrcorner\, \tau^\natural(\phi) = 0\}\,.$$

The following mutually dual local bases of vector fields and forms are adapted to the above splitting

$$(\text{д}_0\,,\ b_i) := (\partial_0 + x^i_0\, \partial_i\,,\ \partial_i - c\,\alpha \tau^\natural_i\, \text{д}_0)$$
$$(\lambda^0\,,\ \theta^i) := (d^0 + c\,\alpha\, \tau^\natural_i\, \theta^i\,,\ d^i - x^i_0\, d^0)\,.$$

We observe that

$$\lambda^0 = c\,\alpha \tau^\natural\,.$$

With reference to a natural base, it is convenient to introduce the matrices

$$h_{i\mu} := g_{i\mu} - c^2 \tau^\natural_i \tau^\natural_\mu \,, \qquad h^{i\mu} := g^{i\mu} - x^i_0 g^{0\mu} \,,$$

and, with reference to an adapted base, the mutually inverse matrices

$$g^\perp_{ij} := g_{ij} - c^2 \tau^\natural_i \tau^\natural_j \,, \qquad g^{ij}_\perp := g^{ij} - g^{i0} x^j_0 - g^{j0} x^i_0 + g^{00} x^i_0 x^j_0 \,.$$

Then, we obtain the following coordinate expressions

$$g^\flat \circ b_i = g^\perp_{ij} \theta^j = h_{i\mu} d^\mu \,, \qquad g^\sharp \circ \theta^i = g^{ij}_\perp b_j = h^{i\mu} \partial_\mu \,.$$

Moreover, the parallel and orthogonal projections

$$\lambda : UM \underset{M}{\times} TM \to T^{\|}M \,, \qquad \theta = 1_M - \lambda : UM \underset{M}{\times} TM \to T^\perp M$$

have the coordinate expressions

$$\lambda = \lambda^0 \otimes д_0 \,, \qquad \theta = h^{i\mu} h_{i\nu} d^\nu \otimes \partial_\mu \,.$$

Additionally, we have a natural linear fibred isomorphism over UM

$$v^\perp : VUM \to \mathbb{T}^* \otimes T^\perp M \,,$$

with coordinate expression

$$v^\perp = c\,\alpha\, d^i_0 \otimes b_i \,, \qquad v^{\perp\,-1} = \frac{1}{c\,\alpha} \theta^i \otimes \partial^0_i \,.$$

1.4. **Connections, 2–forms and 2–vectors.** A linear connection of spacetime induces naturally on the phase space a connection, a second order connection, a 2–form and a 2–vector, which fulfill certain relations.

A linear connection K on the vector bundle $\pi_M : TM \to M$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$K : TM \to T^*M \underset{TM}{\otimes} TTM \,, \qquad \nu[K] : TM \to T^*TM \underset{TM}{\otimes} TM \,,$$

respectively, with coordinate expressions

$$K = d^\phi \otimes (\partial_\phi + K_\phi{}^\mu{}_\psi \, \dot{x}^\psi \, \dot{\partial}_\mu) \,, \qquad \nu[K] = (\dot{d}^\mu - K_\phi{}^\mu{}_\psi \, \dot{x}^\psi \, d^\phi) \otimes \partial_\mu \,,$$

where $K_\phi{}^\mu{}_\psi \in \mathcal{F}(M)$ and $(x^\phi, \dot{x}^\phi)$ is the induced coordinate chart on TM.

We observe that a linear connection $\nu[K]$ on $TM \to M$ induces a linear connection $\nu[K'] : T(\mathbb{T}^* \otimes TM) \to \mathbb{T}^* \otimes TM$ on the vector bundle $\mathbb{T}^* \otimes TM \to M$, with coordinate expression

$$\nu[K'] = u^0 \otimes (\dot{d}^\mu_0 - K_\phi{}^\mu{}_\psi \, \dot{x}^\psi_0 \, d^\phi) \otimes \partial_\mu \,,$$

where u^0 is a base of $\mathbb{T}^*$ and $(x^\phi, \dot{x}^\phi_0)$ denotes the induced chart on $\mathbb{T}^* \otimes TM$.

A connection Γ on UM can be expressed, equivalently, by a tangent valued form, or by a vector valued form

$$\Gamma : UM \to T^*M \underset{UM}{\otimes} TUM \,, \qquad v^\perp \circ \nu[\Gamma] : UM \to T^*UM \underset{UM}{\otimes} (\mathbb{T}^* \otimes T^\perp M) \,,$$

with coordinate expressions

$$\Gamma = d^\phi \otimes (\partial_\phi + \Gamma_\phi{}^i_0 \, \partial^0_i) \,, \qquad v^\perp \circ \nu[\Gamma] = c\,\alpha\, (d^i_0 - \Gamma_\phi{}^i_0 \, d^\phi) \otimes b_i \,,$$

where $\Gamma_\phi{}^i_0 \in \mathcal{F}(UM)$.

For any linear connection K on TM the map

$$\nu[\Gamma] = v^{\perp^{-1}} \circ \theta \circ \nu[K'] \circ T д$$

turns out to be a connection on the bundle $UM \to M$ with coordinate expression

$$\Gamma_{\phi}{}^{i}_{0} = K_{\phi}{}^{i}{}_{j}\, x^{j}_{0} + K_{\phi}{}^{i}{}_{0} - x^{i}_{0}\,(K_{\phi}{}^{0}{}_{j}\, x^{j}_{0} + K_{\phi}{}^{0}{}_{0})\,.$$

A connection Γ on UM and the metric g yield the scaled 2–form on UM

$$\omega(g, \Gamma) := (v^{\perp} \circ \nu[\Gamma]) \bar{\wedge} \theta : UM \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \Lambda^2 T^* UM\,,$$

where $\bar{\wedge}$ denotes the wedge product followed by the contraction via the metric g.

We have the coordinate expression

$$\omega(g, \Gamma) = c\,\alpha\, h_{i\mu}\,(d^{i}_{0} - \Gamma_{\phi}{}^{i}_{0}\, d^{\phi}) \wedge d^{\mu}\,.$$

Moreover, a connection Γ on UM and the metric g yield the scaled 2–vector on UM

$$\tilde{\omega}(g, \Gamma) := \Gamma\, \tilde{\bar{\wedge}}\, v^{\perp^{-1}} : UM \to (\mathbb{T} \otimes \mathbb{L}^{*2}) \otimes \Lambda^2 TUM\,,$$

where $\tilde{\bar{\wedge}}$ denotes the wedge product follwed by the contraction via the dual metric

$$\tilde{g} : M \to \mathbb{L}^{*2} \otimes TM \underset{M}{\otimes} TM\,.$$

We have the coordinate expression

$$\tilde{\omega}(g, \Gamma) = \frac{1}{c\,\alpha}\, h^{i\lambda}\,(\partial_{\lambda} + \Gamma_{\lambda}{}^{j}_{0}\, \partial^{0}_{j}) \wedge \partial^{0}_{i}\,.$$

Let us denote by $U_2 M \subset J_2(M, 1)$ the subspace of all 2–jets of motions.

A *second order connection* on M is defined to be a section

$$\gamma : UM \to U_2 M\,,$$

which can be characterized as a scaled vector field

$$\gamma : UM \to \mathbb{T}^* \otimes TUM\,,$$

such that

$$T\pi^{1}_{0} \circ \gamma = д : UM \to \mathbb{T}^* \otimes TM\,.$$

Hence, we obtain

$$\gamma \lrcorner\, \tau^{\natural} = 1 : UM \to \mathbb{T}^* \otimes \mathbb{T}\,.$$

The coordinate expression of γ is of the type

$$\gamma = c\,\alpha\,(\partial_0 + x^{i}_{0}\, \partial_i + \gamma_{0}{}^{i}_{0}\, \partial^{0}_{i})\,, \qquad \gamma_{0}{}^{i}_{0} \in \mathcal{F}(UM)\,.$$

There is a unique second order connection γ such that

$$\gamma \lrcorner\, \omega(g, \Gamma) = 0\,;$$

namely, the second order connection

$$\gamma = д \lrcorner\, \Gamma : UM \to \mathbb{T}^* \otimes TUM$$

with coordinate expression

$$\gamma_{0}{}^{i}_{0} = \Gamma_{k}{}^{i}_{0}\, x^{k}_{0} + \Gamma_{0}{}^{i}_{0}\,.$$

1.5. **Gravitational objects.** In the above section we have analysed some possible objects on spacetime. Now, we start by considering the objects of this kind that come from the Lorentz metric.

The metric g yields the *gravitational* connection on TM,

$$K^\natural := \kappa\,,$$

where κ is the Levi–Civita connection.

Hence, according to the above discussion, we obtain the *gravitational* connection on UM and the *gravitational* second order connection on UM

$$\nu[\Gamma^\natural] := v^{\perp^{-1}} \circ \theta \circ \nu[K^{\natural}{}'] \circ T\text{д}\,, \qquad \gamma^\natural := \text{д} \lrcorner \Gamma^\natural\,.$$

Moreover, we obtain the *gravitational* 2–form and the the *gravitational* 2–vector

$$\omega^\natural := \omega(g, \Gamma^\natural)\,, \qquad \tilde{\omega}^\natural := \tilde{\omega}(g, \Gamma^\natural)\,.$$

We observe that the gravitational 2–form is globally exact according to

$$\omega^\natural = c^2 d\tau^\natural$$

and that

$$c^2 \tau^\natural \wedge \omega^\natural \wedge \omega^\natural \wedge \omega^\natural : UM \to (\mathbb{T}^{*4} \otimes \mathbb{L}^8) \otimes \Lambda^7 T^* UM$$

is a volume form on the phase space. Hence $\omega^\natural$ is a *contact 2–form.*

1.6. **Total connections and 2–forms.** Next, we deform the above geometric structures by a suitable coupling with electromagnetic field.

Thus, we assume the *electromagnetic field* to be a closed scaled 2–form on M

$$F : M \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* M\,.$$

We denote a local potential of F by

$$A : M \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^* M\,.$$

Here, by definition we set $2dA = F$.

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures of the phase space. Namely, we obtain *total* objects by correcting the gravitational objects by an electromagnetic term, in such a way to preserve their original relations.

For this purpose we need a suitable coupling constant. So, we consider a particle with a given mass and charge

$$m \in \mathbb{M}\,, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}\,,$$

and refer to the coupling constant

$$\frac{q}{m} \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{*1/2}\,.$$

We start from the obvious coupling of the electromagnetic field with the gravitational contact 2–form on UM. Accordingly, we define the *scaled total 2–form* to be

$$\omega := \omega^\natural + \frac{q}{2mc} F : UM \to (\mathbb{T}^* \otimes \mathbb{L}^2) \otimes \Lambda^2 T^* UM\,.$$

Clearly, the total 2–form is closed

$$d\omega = 0$$

and we obtain a local potential of ω of the type

$$c^2\tau := c^2\tau^{\natural} + \frac{q}{mc} A \,.$$

Moreover, we obtain the scaled volume form on the phase space

$$c^2\tau \wedge \omega \wedge \omega \wedge \omega = c^2\tau^{\natural} \wedge \omega^{\natural} \wedge \omega^{\natural} \wedge \omega^{\natural} \,.$$

There is a unique second order connection γ, called the *total second order connection*, such that [7]

$$\gamma \lrcorner \, \omega = 0 \,,$$

namely, the second order connection given by

$$\gamma = \gamma^{\natural} + \gamma^{e} \,,$$

where

$$\gamma^{e} = -\frac{q}{mc} v^{\perp -1} \circ g^{\sharp} \circ (д \lrcorner \, F) \,,$$

is just the *Lorentz force* with coordinate expression

$$\gamma^{e} = -\frac{q}{mc} \, h^{i\mu} (F_{0\mu} + F_{j\mu} \, x^{j}_{0}) \, \partial^{0}_{i} \,.$$

There is a unique connection Γ, called the *total connection*, such that [7]

$$(v^{\perp} \circ \nu_{\Gamma}) \bar{\wedge} \theta = \omega \,,$$

namely, the connection given by

$$\Gamma = \Gamma^{\natural} + \Gamma^{e} \,,$$

where

$$\Gamma^{e} = -\frac{q}{2mc} v^{\perp -1} \circ g^{\sharp 2} \circ (F - 4\tau^{\natural} \wedge (д \lrcorner \, F)) \,,$$

i.e. , in coordinates

$$\Gamma^{e} = \frac{q}{2mc^2\alpha} h^{i\rho} \big(- F_{\lambda\rho} + c\,\alpha \tau^{\natural}_{\lambda} (F_{\rho 0} + F_{\rho r} x^{r}_{0}) \big) \, d^{\lambda} \otimes \partial^{0}_{i} \,.$$

Actually, we obtain

$$д \lrcorner \, \Gamma = \gamma \,.$$

The total phase connection Γ yields the *scaled total* 2*–vector*

$$\tilde{\omega} := \Gamma \, \tilde{\bar{\wedge}} \, v^{\perp -1} \,,$$

given by

$$\tilde{\omega} = \tilde{\omega}^{\natural} + \Gamma^{e} \, \tilde{\bar{\wedge}} \, v^{\perp -1} \,,$$

i.e. , in coordinates by

$$\Gamma^{e} \, \tilde{\bar{\wedge}} \, v^{\perp -1} =$$

$$= -\frac{q}{2mc^3\alpha^2} \, g^{ih}_{\perp} \, g^{jk}_{\perp} \big(F_{kh} + \alpha\, c \, (\tau^{\natural}_{h} \, (F_{0k} + F_{rk} x^{r}_{0}) - \tau^{\natural}_{k} \, (F_{0h} + F_{rh} x^{r}_{0})) \big) \, \partial^{0}_{i} \wedge \partial^{0}_{j} \,.$$

We can normalize the scaled total 2–form ω and 2–vector $\tilde{\omega}$ and obtain the true total 2–form and 2–vector

$$\Omega := \frac{m}{\hbar}\omega : UM \to \Lambda^2 T^* UM \,, \qquad \tilde{\Omega} := \frac{\hbar}{m}\tilde{\omega} : UM \to \Lambda^2 TUM \,.$$

From now on we shall refer to the total objects

$$\Gamma \,, \qquad \gamma \,, \qquad \Omega \,, \qquad \tilde{\Omega} \,.$$

2. Tangent space of phase space

In this section we analyse the splitting of the tangent space of the phase space and the musical maps induced by the total contact 2–form.

2.1. Splitting of the tangent space of phase space.

We start with the splittings of the sheaves $\mathcal{T}(UM)$ of vector fields and $\mathcal{T}^*(UM)$ of 1-forms on UM induced by the 1-form $\tau^\natural$ and the second order connection γ.

We introduce the subsheaves

$$\mathcal{T}^{\perp}_{\tau^\natural} \subset \mathcal{T}(UM) \,, \qquad \mathcal{T}^{\gamma}_{\perp} \subset \mathcal{T}^*(UM) \,,$$

defined by

$$\mathcal{T}^{\perp}_{\tau^\natural} := \{X \in \mathcal{T}(UM) \ \mid \ X \lrcorner \tau^\natural = 0\} \,, \qquad \mathcal{T}^{\gamma}_{\perp} := \{\phi \in \mathcal{T}^*(UM) \ \mid \ \gamma \lrcorner \phi = 0\} \,,$$

and the subsheaves

$$\langle \frac{\gamma}{c\,\alpha} \rangle \subset \mathcal{T}(UM) \,, \qquad \langle c\,\alpha\,\tau^\natural \rangle \subset \mathcal{T}^*(UM)$$

generated by γ and $\tau^\natural$ by means of $\mathbb{T}$- and $\mathbb{T}^*$- valued functions, respectively.

We have the natural splittings

$$\mathcal{T}(UM) = \langle \frac{\gamma}{c\,\alpha} \rangle \oplus \mathcal{T}^{\perp}_{\tau^\natural} \,, \qquad \mathcal{T}^*(UM) = \langle c\,\alpha\,\tau^\natural \rangle \oplus \mathcal{T}^{\gamma}_{\perp} \,,$$

given by

$$X = \gamma(X \lrcorner \tau^\natural) + \big(X - \gamma(X \lrcorner \tau^\natural)\big) \,, \qquad \phi = \tau^\natural(\gamma \lrcorner \phi) + \big(\phi - \tau^\natural(\gamma \lrcorner \phi)\big) \,,$$

and the natural sheaf isomorphisms

$$\langle \frac{\gamma}{c\,\alpha} \rangle^* \simeq \langle c\,\alpha\,\tau^\natural \rangle \,, \qquad \mathcal{T}^{\perp *}_{\tau^\natural}(UM) \simeq \mathcal{T}^{\gamma}_{\perp}(UM) \,.$$

We have the coordinate expressions

$$\gamma(X \lrcorner \tau^\natural) = c\,\alpha\,\tau^\natural_\lambda\, X^\lambda\,(\partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i) \,,$$
$$\tau^\natural(\gamma \lrcorner \phi) = c\,\alpha\,(\phi_0 + \phi_p\,x^p_0 + \phi^0_p\,\gamma_0{}^p_0)\tau^\natural_\lambda d^\lambda \,.$$

Remark 2.1. Obviously, we have

$$\mathcal{V}(UM) \subset \mathcal{T}^{\perp}_{\tau^\natural}(UM) \,, \qquad T\pi^1_0(\mathcal{T}^{\perp}_{\tau^\natural}(UM)) = \mathcal{T}^{\perp}M \,,$$

where $\mathcal{V}(UM)$ denotes the sheaf of vertical vector fields of the bundle $UM \to M$. □

2.2. **Adapted bases.** We can introduce local bases of vector fields and 1-forms on UM, which are suitable for coordinate computations.

Lemma 2.1. *The following mutually dual local bases of vector fields and 1-forms on UM are adapted to the splittings induced by γ and $\tau^\natural$*

$$(B_0\,,\,B_i\,,\,B^0_i) = \big(\partial_0 + x^j_0\partial_j + \gamma_0{}^i_0\,\partial^0_i\,,\quad \partial_i - c\,\alpha\,\tau^\natural_i\,(\partial_0 + x^j_0\,\partial_j + \gamma_0{}^j_0\,\partial^0_j)\,,\quad \partial^0_i\big)\,,$$

$$(\lambda^0\,,\,\theta^i\,,\,\theta^i_0) = \big(d^0 - c\,\alpha\,\tau^\natural_i\,\theta^i\,,\quad d^i - x^i_0\,d^0\,,\quad d^i_0 - \gamma_0{}^i_0\,d^0\big)\,. \qquad \square$$

Remark 2.2. If $X = X^0\,B_0 + X^i\,B_i + X^i_0\,B^0_i \in \mathcal{T}(UM)$, then

$$\gamma(X \lrcorner\, \tau^\natural) = X^0\,B_0 \in \langle\gamma\rangle \qquad X - \gamma(X \lrcorner\, \tau^\natural) = X^i\,B_i + X^i_0\,B^0_i \in \mathcal{T}^\perp_{\tau^\natural}(UM)\,.$$

If $\phi = \phi_0\,\theta^0 + \phi_i\,\theta^i + \phi^0_i\,\theta^i_0 \in \mathcal{T}^*(UM)$, then

$$\tau^\natural(\gamma \lrcorner\, \phi) = \phi_0\,\theta^0 \in \langle\tau^\natural\rangle \qquad \phi - \tau^\natural(\gamma \lrcorner\, \phi) = \phi_i\,\theta^i + \phi^0_i\,\theta^i_0 \in \mathcal{T}^\gamma_\perp(UM)\,. \qquad \square$$

Remark 2.3. The coordinate expressions of the scaled 2–form $\omega(g,\Gamma)$ and the scaled 2–vector $\tilde\omega(g,\Gamma)$ in an adapted base are

$$\omega(g,\Gamma) = c\,\alpha\,g^\perp_{ij}\,(\theta^i_0 - \Gamma_h{}^i_0\,\theta^h) \wedge \theta^j\,, \qquad \tilde\omega(g,\Gamma) = \frac{1}{c\,\alpha}\,g^{ij}_\perp\,(B_j + \Gamma_j{}^h_0\,B^0_h) \wedge B^0_i\,. \qquad \square$$

2.3. **Musical mappings.** The total 2–form Ω and the total 2–vector $\tilde\Omega$ yield mutual inverse isomorphisms between the contravariant and covariant orthogonal components of the above splitting.

Lemma 2.2. *The total sections*

$$\Omega : UM \to \Lambda^2 T^*UM\,, \qquad \tilde\Omega : UM \to \Lambda^2 TUM$$

yield the linear sheaf epimorphisms

$$\mathcal{T}(UM) \to \mathcal{T}^\gamma_\perp(UM) : X \mapsto i[X]\Omega\,, \qquad \mathcal{T}^*(UM) \to \mathcal{T}^\perp_{\tau^\natural}(UM) : \phi \mapsto i[\phi]\tilde\Omega\,,$$

with coordinate expressions, in a natural base,

$$i[X]\Omega = \frac{m\,c\,\alpha}{\hbar}\big((h_{i\lambda}X^i_0 + h_{i\mu}\,X^\mu\,\Gamma_\lambda{}^i_0 - h_{i\lambda}\Gamma_\phi{}^i_0\,X^\phi)\,d^\lambda - h_{i\mu}\,X^\mu\,d^i_0\big)\,,$$

$$i[\phi]\tilde\Omega = \frac{\hbar}{m\,c\,\alpha}\big(-h^{j\lambda}\,\phi^0_j\,\partial_\lambda + (h^{i\lambda}\,\phi_\lambda + h^{i\lambda}\,\Gamma_\lambda{}^j_0\,\phi^0_j - h^{j\lambda}\,\Gamma_\lambda{}^i_0\,\phi^0_j)\,\partial^0_i\big)\,. \qquad \square$$

Theorem 2.1. *The above maps restrict to the mutually inverse linear sheaf isomorphisms*

$$\Omega^\flat : \mathcal{T}^\perp_{\tau^\natural}(UM) \to \mathcal{T}^\gamma_\perp(UM)\,, \qquad \Omega^\sharp : \mathcal{T}^\gamma_\perp(UM) \to \mathcal{T}^\perp_{\tau^\natural}(UM)\,,$$

with coordinate expressions, in an adapted base,

$$\Omega^\flat(X) := i[X]\Omega = \frac{m\,c\,\alpha}{\hbar}\big((g^\perp_{ij}\,X^j_0 + g^\perp_{mj}\,\Gamma_i{}^m_0\,X^j - g^\perp_{mi}\,\Gamma_j{}^m_0\,X^j)\,\theta^i - g^\perp_{ij}\,X^j\,\theta^i_0\big)\,,$$

$$\Omega^\sharp(\phi) := i[\phi]\tilde\Omega = \frac{\hbar}{m\,c\,\alpha}\big(-g^{ij}_\perp\,\phi^0_j\,B_i + (g^{ij}_\perp\,\phi_j + g^{mi}_\perp\,\Gamma_m{}^j_0\,\phi^0_j - g^{jm}_\perp\,\Gamma_m{}^i_0\,\phi^0_j)\,B^0_i\big)\,. \qquad \square$$

2.4. **Time scale.** Here, we introduce the notion of time scale, which will play a role in view of Hamiltonian lift of quantisable functions.

We define a *time scale* to be a smooth map

$$t : UM \to \mathbb{T} .$$

Given a time scale t, we introduce the subsheaf

$$\mathcal{T}_t(UM) \subset \mathcal{T}(UM) := \{X \in \mathcal{T}(UM) \mid X \lrcorner \tau^\natural = t\} .$$

Thus, in coordinates, $X \in \mathcal{T}_t(UM)$ if and only if $X^\alpha \tau^\natural_\alpha = t$.

Remark 2.4. Each fibred morphism $\phi : UM \to \mathbb{L}^2 \otimes T^*M$ over M yields the time scale

$$t[\phi] := \frac{д \lrcorner \phi}{c^2}$$

with coordinate expression

$$t[\phi] = \frac{\alpha}{c}(\phi_p \, x^p_0 + \phi_0) . \qquad \square$$

Remark 2.5. For each $X \in \mathcal{T}(UM)$, we can write

$$X \lrcorner \tau^\natural = \frac{д}{c^2} \lrcorner (g^\flat \circ \underline{X}) = t[g^\flat \circ \underline{X}] ,$$

where we set

$$\underline{X} := T\pi^1_0 \circ X : UM \to TM . \qquad \square$$

Remark 2.6. Let $t : UM \to \mathbb{T}$ be a time scale constant on the fibres of $UM \to M$. A vector field $X \in \mathcal{T}_t(UM)$ is projectable on a vector field $\underline{X} \in \mathcal{T}(M)$ if and only if X is vertical and $t = 0$. $\square$

3. Quantisable functions

In this section we describe the Lie algebra of quantisable functions on UM and its lift to the tangent space of spacetime. The quantisable functions will play an important role for quantum vector fields.

3.1. **Quantisable functions.** In the section we exhibit a distinguished sheaf of functions on the phase space which admit a tangent lift. Our construction is based on a projectability criterion as in the Galilei case [1, 2, 3, 4]. This sheaf of functions includes spacetime coordinates, momentum and energy functions for a classical particle. Their lifts are vector fields which are expected to play an important role in the covariant quantisation procedure analogous to that developed in the Galilei case [1, 2, 3, 4]. For this reason, we call these functions quantisable.

Let f be a local function on UM.

We define the γ*–covariant differential* of f to be the section

$$d_\gamma f := df - \tau^\natural(\gamma \lrcorner df) \in \mathcal{T}^{\gamma}_{\perp}(UM) ,$$

with coordinate expression

$$d_\gamma f = (\partial_\lambda f - c\,\alpha\,\tau^\natural_\lambda \left(\partial_0 f + x^p_0 \, \partial_p f + \gamma_0{}^p_0 \, \partial^0_p f)\right) d^\lambda + \partial^0_i f \, d^i_0 .$$

Moreover, we define the sheaf morphism

$$\mathcal{F}(UM) \to \mathcal{T}^{\perp}_{\tau^\natural}(UM) : f \mapsto f^\sharp := \Omega^\sharp(d_\gamma f)\,,$$

with coordinate expression

$$f^\sharp = \frac{\hbar}{m\,c\,\alpha}\big(- h^{j\lambda}\,\partial^0_j f\,\partial_\lambda + (h^{i\lambda}\,\partial_\lambda f + (h^{i\lambda}\,\Gamma_\lambda{}^j_0 - h^{j\lambda}\,\Gamma_\lambda{}^i_0)\,\partial^0_j f)\,\partial^0_i\big)\,.$$

Furthermore, if $t : UM \to \mathbb{T}$ is a time scale, then we define the sheaf morphism

$$\mathcal{F}(UM) \to \mathcal{T}^{t}_{\tau^\natural}(UM) : f \mapsto f^\sharp_t := \gamma(t) + f^\sharp\,,$$

with coordinate expression

$$(3.1)\qquad f^\sharp_t = \big(- \frac{\hbar}{m\,c\,\alpha} h^{j\lambda}\,\partial^0_j f + c\,\alpha\,t\,(\delta^\lambda_0 + \delta^\lambda_i\,x^i_0)\big)\,\partial_\lambda + \big(\frac{\hbar}{m\,c\,\alpha}(h^{i\lambda}\,\partial_\lambda f + (h^{i\lambda}\,\Gamma_\lambda{}^j_0 - h^{j\lambda}\,\Gamma_\lambda{}^i_0)\,\partial^0_j f) + c\,\alpha\,t\,\gamma_0{}^i_0\big)\,\partial^0_i\,.$$

Obviously, we have

$$f^\sharp_0 = f^\sharp\,.$$

If $f \in \mathcal{F}(UM)$ is a function of the type

$$f := -\frac{m}{\hbar}д \lrcorner \phi + f_0\,,$$

where $f_0 \in \mathcal{F}(M)$ and $\phi \in \mathbb{T}^2 \otimes \mathcal{T}^*(M)$, then we set

$$l[f] := \phi\,, \qquad t[f] := \frac{д}{c^2} \lrcorner \phi\,.$$

We have, [8],

Theorem 3.1. *Let $f \in \mathcal{F}(UM)$ and t be a time scale. Then, the vector field $f^\sharp_t$ is projectable on the vector field*

$$X[f] := T\pi^1_0 \circ f^\sharp_t : M \to TM$$

if and only if f is of the type

$$(3.2)\qquad f = -\frac{m}{\hbar}д \lrcorner \phi + f_0\,,$$

where $f_0 \in \mathcal{F}(M)$, $\phi \in \mathbb{L}^2 \otimes \mathcal{T}^*(M)$, and

$$t = t[f] := \frac{д}{c^2} \lrcorner \phi\,.$$

Moreover, if the above conditions are fulfilled, then we obtain

$$X[f] = g^\sharp \circ \phi\,. \qquad \square$$

The functions $f \in \mathcal{F}(UM)$ of the type (3.2) are said to be *quantisable*. The subsheaf of quantisable functions is denoted by

$$\mathcal{Q}(UM) \subset \mathcal{F}(UM)\,.$$

The vector prolongations

$$\mathcal{Q}(UM) \to \mathcal{T}(UM) : f \mapsto f^\sharp_{t[f]}\,, \qquad \mathcal{Q}(UM) \to \mathcal{T}(M) : f \mapsto X[f]\,,$$

are said to be the *phase prolongation* and the *tangent prolongation* of f, respectively.

Corollary 3.1. *The coordinate expression of a quantisable function is of the type*

$$f = -\frac{m\,c}{\hbar}\frac{\phi_j x^j_0 + \phi_0}{\sqrt{g_{00} + 2g_{j0}x^j_0 + g_{ji}x^j_0 x^i_0}} + f_0\,.$$

The corresponding vector fields have the coordinate expressions

$$\begin{aligned} f^\sharp_{t[f]} = g^{\lambda\mu}\,\phi_\mu\,\partial_\lambda + \Big(\frac{\hbar}{m\,c\,\alpha} & h^{i\lambda}\,\partial_\lambda f_0 - h^{i\lambda}(x^k_0\,\partial_\lambda\phi_k + \partial_\lambda\,\phi_0) \\ & + (\partial_\rho g_{0\lambda} + x^k_0\,\partial_\rho g_{k\lambda})(g^{\mu\lambda}\,h^{i\rho} - g^{\mu\rho}\,h^{i\lambda})\,\phi_\mu \\ & + \frac{q}{2mc^2\alpha}(h^{i\lambda}g^{\mu\rho} - h^{i\rho}g^{\mu\lambda})F_{\lambda\rho}\phi_\mu\Big)\,\partial^0_i\,, \\ X[f] = g^{\lambda\mu}\,\phi_\lambda\,\partial_\mu\,. & \end{aligned}$$

Proof. The proof consist in direct difficult computations in coordinates by using (3.1), (3.2) and the formula

$$h^{i\lambda}\,\Gamma_\lambda{}^j_0 - h^{j\lambda}\,\Gamma_\lambda{}^i_0 = (h^{i\rho}\,h^{j\lambda} - h^{i\lambda}\,h^{j\rho})(\partial_\lambda g_{0\rho} + x^r_0\,\partial_\lambda g_{r\rho} + \frac{q}{2mc^2\alpha}F_{\lambda\rho})\,. \qquad \square$$

3.2. **Lie algebra of quantisable functions.** Now we exhibit a Lie bracket for quantisable functions and show that the tangent lift is a Lie algebra homomorphism. For this purpose, we shall use the Lie bracket of 1-forms, [5].

We define the Poisson–Lie bracket

$$\{f_1, f_2\} = i[f^\sharp_1]\,i[f^\sharp_2]\,\Omega = i[f^\sharp_1]d_\gamma f_2$$

with coordinate expression, in a canonical base,

$$\{f_1, f_2\} = -\frac{\hbar}{m\,c\,\alpha}\big(h^{i\lambda}(\partial^0_i f_1\,\partial_\lambda f_2 - \partial_\lambda f_1\,\partial^0_i f_2) + (h^{i\lambda}\,\Gamma_\lambda{}^j_0 - h^{j\lambda}\,\Gamma_\lambda{}^i_0)\,\partial^0_i f_1\,\partial^0_j f_2\big)\,.$$

If $f_1, f_2 \in \mathcal{Q}(UM)$, then, in general, $\{f_1, f_2\} \notin \mathcal{Q}(UM)$.

In order to make the sheaf of quantisable functions a sheaf of Lie algebras, we define the bracket

$$[f_1, f_2] := \{f_1, f_2\} + \gamma(t[f_1]).f_2 - \gamma(t[f_2]).f_1\,.$$

Lemma 3.1. *We have the Lie bracket of* 1*-forms* $\phi, \psi \in \mathbb{L}^2 \otimes \mathcal{T}^*(M)$

$$\{\phi, \psi\}_g = g^\flat([g^\sharp(\phi),\, g^\sharp(\psi)])\,,$$

with coordinate expression

$$\{\phi, \psi\}_g = \big(\partial_\sigma g_{\lambda\rho}\,(g^{\rho\mu}\,g^{\sigma\nu} - g^{\nu\rho}\,g^{\sigma\mu})\,\phi_\mu\,\psi_\nu + g^{\sigma\mu}\,(\phi_\mu\,\partial_\sigma\psi_\lambda - \psi_\mu\,\partial_\sigma\phi_\lambda)\big)\,d^\lambda\,. \qquad \square$$

Theorem 3.2. *If* $f_1, f_2 \in \mathcal{Q}(UM)$, *then*

$$\begin{aligned}[f_1, f_2] = & -\frac{m}{\hbar}\,д \lrcorner \{l[f_1], l[f_2]\}_g + g^\sharp(l[f_1]).f_{20} - g^\sharp(l[f_2]).f_{10} \\ & + \frac{q}{\hbar\,c}F(g^\sharp(l[f_1]), g^\sharp(l[f_2]))\,.\end{aligned}$$

Proof. The proof consists in difficult calculations in coordinates, by taking into account the formula

$$
\begin{aligned}
[f_1, f_2] = -\frac{m\,c\,\alpha}{\hbar}\big((\partial_\sigma g_{k\rho} x_0^k + \partial_\sigma g_{0\rho})(g^{\nu\sigma} g^{\rho\mu} - g^{\nu\rho} g^{\sigma\mu}) l[f_1]_\mu l[f_2]_\nu \\
+ g^{\mu\lambda}(l[f_1]_\lambda \partial_\mu l[f_2]_k - l[f_2]_\lambda \partial_\mu l[f_1]_k) x_0^k + g^{\mu\lambda}(l[f_1]_\lambda \partial_\mu l[f_2]_0 - l[f_2]_\lambda \partial_\mu l[f_1]_0)\big) \\
+ g^{\mu\lambda}(l[f_1]_\lambda \partial_\mu f_{20} - l[f_2]_\lambda \partial_\mu f_{10}) + \frac{q}{\hbar\,c} F_{\lambda\rho} g^{\lambda\mu} l[f_1]_\mu g^{\rho\nu} l[f_2]_\nu\,. \qquad \square
\end{aligned}
$$

Example 3.1. Let $\phi_1, \phi_2 \in \mathbb{L}^2 \otimes \mathcal{T}^*(M)$ and $f_{10}, f_{20} \in \mathcal{F}(M)$. Then

$$
[f_{10}\,, f_{20}] = 0\,, \qquad [-\frac{m}{\hbar}\text{д} \lrcorner \phi_1\,, f_{20}] = g^\sharp(\phi_1).f_{20}\,,
$$

$$
[-\frac{m}{\hbar}\text{д} \lrcorner \phi_1\,, -\frac{m}{\hbar}\text{д} \lrcorner \phi_2] = -\frac{m}{\hbar}\text{д} \lrcorner \{\phi_1, \phi_2\}_g + \frac{q}{\hbar\,c} F(g^\sharp\phi_1, g^\sharp\phi_2)\,. \qquad \square
$$

Theorem 3.3. *$\mathcal{Q}(UM)$ with the bracket $[\,,]$ is a sheaf of $I\!R$–Lie algebras.*

Proof. Let $f_1, f_2, f_3 \in \mathcal{Q}(UM)$. Then, Theorem 3.2 and Example 3.1 yield

$$
\begin{aligned}
[f_1\,, [f_2, f_3]] &= [-\frac{m}{\hbar}\text{д} \lrcorner l[f_1] + f_{10}\,, -\frac{m}{\hbar}\text{д} \lrcorner \{l[f_2], l[f_3]\}_g \\
&\quad + g^\sharp(l[f_2]).f_{30} - g^\sharp(l[f_3]).f_{20} + \frac{q}{\hbar\,c} F(g^\sharp(l[f_2]), g^\sharp(l[f_3]))] \\
&= -\frac{m}{\hbar}\text{д} \lrcorner \{l[f_1], \{l[f_2], l[f_3]\}_g\}_g \\
&\quad + g^\sharp(l[f_1]).g^\sharp(l[f_2]).f_{30} - g^\sharp(l[f_1]).g^\sharp(l[f_3]).f_{20} \\
&\quad - g^\sharp(l[f_2]).g^\sharp(l[f_3]).f_{10} + g^\sharp(l[f_3]).g^\sharp(l[f_2]).f_{10} \\
&\quad + \frac{q}{\hbar\,c} g^\sharp(l[f_1]).F(g^\sharp(l[f_2]), g^\sharp(l[f_3])) \\
&\quad + \frac{q}{\hbar\,c} F(g^\sharp(l[f_1]), [(g^\sharp(l[f_2]), g^\sharp(l[f_3])])\,.
\end{aligned}
$$

The Jacobi identity now follows from the Jacobi identity for the bracket $\{\,,\}_g$ and from the closure of F. $\square$

Clearly we have

Corollary 3.2. *The tangent prolongation of quantisable functions is an $I\!R$–Lie algebra epimorphism, i.e.*

$$
X[[f_1, f_2]] = [X[f_1]\,, X[f_2]]\,. \qquad \square
$$

4. Lie algebra of quantum vector fields

In this section, we analyse the Lie algebra of quantum vector fields on the quantum bundle and exhibit a Lie algebra isomorphism between quantisable functions and quantum vector fields.

4.1. **Complex line bundle.** We start with some basic properties of a complex line bundle.

Let us consider a *Hermitian line bundle* $\pi : F \to B$, i.e. a complex vector bundle with 1-dimensional fibres, equipped with a Hermitian product $h : F \times_B F \to \mathbb{C}$.

We shall refer to *normal bases*, i.e. to (local) sections $b : B \to F$ such that $h(b, b) = 1$ and to the associated coordinate functions on the fibres $z : F \to \mathbb{C}$.

Accordingly, given the sections $\Psi, \Phi : B \to F$, we write

$$\Psi = \psi b, \quad \Phi = \varphi b\,, \qquad h(\Psi, \Phi) = \bar{\psi}\varphi\,.$$

The Liouville vector field $\text{и} : F \to VF = F \times_B F$ will be identified with $\text{и} = \mathrm{id}_F : B \to F^* \otimes_B F$. In a normal base $\text{и} = z \otimes b$.

A projectable vector field $X : F \to TF$ is said to be *Hermitian* if it is real linear over its projection $\underline{X} : B \to TB$ and preserves the Hermitian product. A Hermitian vector field turns out to be also complex linear and its coordinate expression in a normal coordinate chart is of the type

$$X = X^\lambda \partial_\lambda + i\, f\, \text{и}\,, \qquad f,\, X^\lambda \in \mathcal{F}(B)\,.$$

Lemma 4.1. *We have the natural sheaf isomorphism $\mathcal{F}(B) \to \mathcal{V}(F)$ given by*

$$f \mapsto f^v = i\, f\, \text{и}\,. \qquad \square$$

Let

$$c : F \to T^*B \underset{F}{\otimes} TF$$

be a connection on F and X be a projectable vector field. The covariant differential of X with respect to c is defined to be

$$d[c]X := [c, X] : F \to T^*B \underset{F}{\otimes} VF\,,$$

where $[\,,]$ is the Frölicher-Nijenhuis bracket. In coordinates if

$$c = d^\lambda \otimes \partial_\lambda + c_\lambda\, d^\lambda \otimes b\,, \qquad X = X^\lambda \partial_\lambda + X^z\, b\,,$$

where $c_\lambda, X^z \in \mathbb{C} \otimes \mathcal{F}(F)$, $X^\lambda \in \mathcal{F}(B)$, then

$$d[c]X = (-\partial_\lambda X^\mu c_\mu - \partial_\mu c_\lambda\, X^\mu + \partial_\lambda X^z + c_\lambda \partial_z X^z - \partial_z c_\lambda\, X^z)\, d^\lambda \otimes b\,.$$

Moreover,

$$d[c](c(\underline{X})) = -2\underline{X} \lrcorner R[c]\,,$$

where $R[c] : F \to \Lambda^2 T^*B \otimes_F VF$ is the curvature of c.

Lemma 4.2. *Let c be a linear connection on F and $X = i\, f\, \text{и}$ be a Hermitian vertical vector field on F. Then*

$$d[c]X = i\, df \otimes \text{и}\,. \qquad \square$$

4.2. **Quantum bundle and quantum connection.** We assume the *quantum bundle* to be a Hermitian line bundle over spacetime

$$\pi : Q \to M\,.$$

Moreover, we consider the pullback bundle of the quantum bundle with respect to the phase space

$$\pi^{\uparrow} : Q^{\uparrow} := UM \underset{M}{\otimes} Q \to UM\,;$$

here we regard the bundle UM as a bundle of parameters, whose sections $o : M \to UM$ are interpreted as classical observers.

A connection

$$\text{ч} : Q^{\uparrow} \to T^{*}UM \underset{UM}{\otimes} TQ^{\uparrow}$$

is said to be a *quantum connection* if, [2, 3, 13],

(1) ч is Hermitian,

(2) ч is *universal*,

(3) the curvature of ч is given by

$$R[\text{ч}] = i\,\Omega \otimes \text{и} : Q^{\uparrow} \to \Lambda^{2}T^{*}UM \underset{UM}{\otimes} Q^{\uparrow}\,.$$

For the definition of universal connection see [3], [10]. Very briefly: if $\{\xi[o]\}$ is a system of Hermitian connections on the bundle $\pi : Q \to M$, parametrised by the family of observers, then there exists a unique connection ч on the bundle $Q^{\uparrow} \to UM$ such that, for each observer $o : M \to UM$, the pullback $o^{*}\text{ч}$ equals $\xi[o]$. This connection is said to be *universal*. In coordinates, a universal connection is characterised by the condition $\text{ч}^{0}_{i} = 0$.

The coordinate expression of a universal Hermitian connection ч on $Q^{\uparrow}$ is of the type

$$\text{ч} = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i} + i\,\text{ч}_{\lambda} d^{\lambda} \otimes \text{и}\,, \qquad \text{ч}_{\lambda} \in \mathcal{F}(UM)\,. \tag{4.1}$$

The coordinate expression of the curvature of such a connection ч is

$$R[\text{ч}] = i\,(d\text{ч}_{\lambda}) \wedge d^{\lambda} \otimes \text{и}\,.$$

Hence, the coordinate expression of a quantum connection is of the type

$$\text{ч} = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i} + i\,\frac{m}{\hbar}(c^{2}\tau^{\natural}_{\lambda} + \frac{q}{mc}A_{\lambda})\, d^{\lambda} \otimes \text{и}\,, \tag{4.2}$$

where $A = A_{\lambda} d^{\lambda}$ is a local particular electromagnetic potential (determined by ч).

A pair $(Q, \text{ч})$ is said to be a *quantum structure*.

4.3. **Upper quantum vector fields.** Next we classify the vector fields on the pull-backed quantum bundle, which preserve the quantum structure.

We define an *upper quantum vector field* to be a vector field

$$X^{\uparrow} : Q^{\uparrow} \to TQ^{\uparrow}$$

which

(1) projects onto a vector field

$$\underline{X} : UM \to TUM\,,$$

(2) is Hermitian complex linear over $\underline{X}$,
(3) its covariant differential with respect to the quantum connection ч is proportional to $\tau^\natural$ (with a coefficient in $\mathbb{T}^*$), i.e.

$$d[ч]X^\uparrow \in \langle \tau^\natural \rangle \subset \mathcal{T}^*(UM)\,,$$

where $\langle \tau^\natural \rangle$ is the 1-dimensional space generated by $\tau^\natural$ with coefficients in $\mathbb{T}^*$.

Remark 4.1. The condition (3) can be reformulated as

$$(3') \qquad \tau^\natural(\gamma \lrcorner\, d[ч]X^\uparrow) = d[ч]X^\uparrow\,. \qquad \square$$

The sheaf of upper quantum vector fields will be denoted by $\mathcal{Q}(Q^\uparrow) \subset \mathcal{T}(Q^\uparrow)$

Given a time scale $t : UM \to \mathbb{T}$, we have the subsheaf $\mathcal{Q}_t(Q^\uparrow) \subset \mathcal{Q}(Q^\uparrow)$ of upper quantum vector fields whose time components equal to t, i.e. such that $\underline{X} \lrcorner\, \tau^\natural = t$.

Theorem 4.1. *For each time scale $t : UM \to \mathbb{T}$, we have an $\mathbb{R}$-linear sheaf isomorphism*

$$q^\uparrow_t : \mathcal{F}(UM) \to \mathcal{Q}_t(Q^\uparrow) : f \mapsto X^\uparrow_{f,t}$$

given by

$$X^\uparrow_{f,t} = ч(f^\natural_t) + i\, f\, и\,.$$

Proof. Let us suppose that $X^\uparrow$ be an upper quantum vector field with a given time component t. Let us decompose $X^\uparrow$ into its horizontal and vertical components with respect to the quantum connection ч, i.e.

$$X^\uparrow = ч(\underline{X}) + \nu[ч](X^\uparrow)\,.$$

Linearity of ч and conditions (1) and (2) imply that $\nu[ч](X^\uparrow)$ is Hermitian, i.e. it is of the type

$$\nu[ч](X^\uparrow) = i\, f\, и\,, \qquad f \in \mathcal{F}(UM)\,.$$

From Lemma 4.2 we have

$$d[ч]X^\uparrow = d[ч](ч(\underline{X})) + i\, df \otimes и$$

and from (3')

$$d[ч](ч(\underline{X})) + i\, df \otimes и = \tau^\natural(\gamma \lrcorner\, d[ч](ч(\underline{X})) + i\, \tau^\natural(\gamma \lrcorner\, df) \otimes и\,.$$

But, in virtue of a general formula, we can write

$$d[ч](ч(\underline{X})) = -2\, \underline{X} \lrcorner\, R[ч] = -2\, i\, \underline{X} \lrcorner\, \Omega \otimes и = -i\, \Omega^\flat_t(\underline{X}) \otimes и\,,$$

and we have from $\gamma \lrcorner\, \Omega = 0$

$$df - \tau^\natural(\gamma \lrcorner\, df) = \Omega^\flat_t(\underline{X})\,,$$

i.e.

$$\underline{X} = f^\natural_t\,,$$

and

$$X^\uparrow = ч(f^\natural_t) + i f и\,.$$

On the other hand, we have to prove that for any $f \in \mathcal{F}(UM)$ and a time scale t, the vector field

$$X^{\uparrow}_{f,t} = \text{ч}(f^{\sharp}_t) + i f \text{и}$$

is an upper quantum vector field with the time component t. Clearly this vector field is linear over its projection $f^{\sharp}_t$ and its time component is t. By Remark 4.1 it is sufficient to prove (3'). We have

$$\begin{aligned} d[\text{ч}](\text{ч}(f^{\natural}_t) + i\, f\, \text{и}) &= -2i\, f^{\natural}_t \lrcorner\, \Omega \otimes \text{и} + i\, df \otimes \text{и} \\ &= -i\, \Omega^{\flat}(f^{\natural}_t) \otimes \text{и} + i\, df \otimes \text{и} \\ &= -i\,(df - \tau^{\natural}(\gamma \lrcorner\, df)) \otimes \text{и} + i\, df \otimes \text{и} \\ &= i\, \tau^{\natural}(\gamma \lrcorner\, df) \otimes \text{и}\,. \end{aligned}$$

The result now follows from $\gamma \lrcorner\, \tau^{\natural} = 1$. □

From the coordinate expressions for ч and $f^{\sharp}_t$ we have the coordinate expression

(4.3) $$\begin{aligned} X^{\uparrow}_{f,t} = \text{ч}(f^{\sharp}_t) + i\, f\, \text{и} = \big(-\frac{\hbar}{m\, c\, \alpha} h^{j\lambda}\, \partial^0_j f + c\, \alpha\, t(\delta^{\lambda}_0 + \delta^{\lambda}_i\, x^i_0)\big)\, \partial_\lambda \\ + \big(\frac{\hbar}{m\, c\, \alpha}(h^{i\lambda}\, \partial_\lambda f + (h^{i\lambda}\, \Gamma_{\lambda 0}{}^{j} - h^{j\lambda}\, \Gamma_{\lambda 0}{}^{i})\, \partial^0_j f) + c\, \alpha\, t\, \gamma^i_{00}\big)\, \partial^0_i \\ + i\left(f + \frac{mc^2}{\hbar} t + \frac{q\alpha}{\hbar} t(A_0 + A_p x^p_0) - \frac{q}{mc^2\alpha} h^{j\mu} A_\mu \partial^0_j f\right) \text{и}\,. \end{aligned}$$

4.4. **Quantum vector fields.** Let us consider the natural fibred epimorphism $pr : TQ^{\uparrow} \to TQ$. A vector field $X^{\uparrow} : Q^{\uparrow} \to TQ^{\uparrow}$ is said to be *projectable* on Q if

$$pr \circ X^{\uparrow} : UM \underset{M}{\times} Q \to TQ$$

can be factorised through the vector field

$$X := pr \circ X^{\uparrow} : Q \to TQ\,.$$

In view of possible quantum operators analogously to the scheme of the Galilei case, [3], we are looking for projectable upper quantum vector fields.

Theorem 4.2. *If $X^{\uparrow}_{f,t}$ is an upper quantum vector field, then the following conditions are equivalent:*

i) *$X^{\uparrow}_{f,t}$ is projectable on Q;*

ii) *f is a quantisable function and the time component is $t = t[f]$.*

Proof. i) ⇒ ii). Because of Theorem 4.1, if $X^{\uparrow}_{f,t}$ is projectable on Q, then $f^{\sharp}_t$ is projectable on M hence, by Theorem 3.1, f is a quantisable function and $t = t[f]$.

ii) $\Rightarrow$ i). Given a quantisable function $f = -\frac{m}{\hbar}д \lrcorner \phi + f_0$ we have the corresponding upper quantum vector field with coordinate expression

$$
\begin{aligned}
X^{\uparrow}_{f,t[f]} = g^{\lambda\mu}\,\phi_\mu\,\partial_\lambda + \Big(&\frac{\hbar}{m\,c\,\alpha} h^{i\lambda}\,\partial_\lambda f_0 - h^{i\lambda}(x^k_0\,\partial_\lambda\phi_k + \partial_\lambda\,\phi_0) \\
&+ (\partial_\rho g_{0\lambda} + x^k_0\,\partial_\rho g_{k\lambda})(g^{\mu\lambda}\,h^{i\rho} - g^{\mu\rho}\,h^{i\lambda})\,\phi_\mu \\
&+ \frac{q}{2mc^2\alpha}(h^{i\lambda}g^{\mu\rho} - h^{i\rho}g^{\mu\lambda})F_{\lambda\rho}\phi_\mu\Big)\,\partial^0_i \\
&+ i\,\Big(f_0 + \frac{q}{\hbar\,c} g^{\lambda\mu}A_\lambda\phi_\mu\Big)\,и\,,
\end{aligned}
$$

It is easy to see that this vector field is projectable on Q. □

A *quantum vector field* is defined to be the vector field

$$Y[f] : Q \to TQ$$

associated with an upper quantum vector field $X^{\uparrow}_{f,t}$ projectable on Q. Clearly, $Y[f]$ is projectable on the tangent lift $X[f]$.

Corollary 4.1. *The coordinate expression of the quantum vector field associated with a quantisable function*

$$f = -\frac{m}{\hbar}д \lrcorner \phi + f_0$$

is

$$Y[f] = g^{\lambda\mu}\,\phi_\lambda\,\partial_\mu + i\,\Big(f_0 + \frac{q}{\hbar\,c} g^{\lambda\mu}A_\lambda\phi_\mu\Big)\,и\,. \qquad \square \tag{4.4}$$

The quantum vector fields constitute a sheaf, which is denoted by

$$\mathcal{Q}(Q) \subset \mathcal{T}(Q)\,.$$

Corollary 4.2. *The map*

$$q : \mathcal{Q}(UM) \to \mathcal{Q}(Q)\,, \qquad f \mapsto Y[f]$$

is a sheaf linear isomorphisms.

Proof. This follows from Theorem 4.2. □

Theorem 4.3. *The sheaf $\mathcal{Q}(Q)$ is closed under the Lie bracket and the map*

$$q : \mathcal{Q}(UM) \to \mathcal{Q}(Q)\,, \qquad f \mapsto Y[f]$$

is an isomorphism of sheaves of Lie algebras.
Namely, for each $k \in I\!R$, $f_1, f_2 \in \mathcal{Q}(UM)$, we have

$$Y[kf] = kY[f]\,, \quad Y[f_1 + f_2] = Y[f_1] + Y[f_2]\,, \quad [Y[f_1]\,, Y[f_2]] = Y[[f_1\,,\,f_2]]\,.$$

Proof. It is a consequence of (4.4). □

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