Complex Arrangements and Derivations

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Abstract

Let A be a finite family of proper affine subspaces $h \subset \mathbb{R}^n$ and $M(A) = \mathbb{R}^n \setminus \bigcup_{h \in A} h$. The union $\bigcup_{h \in A} h$ has the homotopy type of the geometric realisation $|\Delta L^\bullet|$ of the order–complex $\Delta L^\bullet$ of $L^\bullet := L - \{\hat{0}\}$. Cohomology as well as homotopy properties of $M(A)$ are discussed. The order–complex $\Delta L^\bullet$ is studied from an algebraic point of view.

Let A be a finite family of proper affine subspaces $h \subset \mathbb{R}^n$ and denote by $L(A)$ the join-semilattice of all non-empty intersections of subfamilies of A partially ordered by reversed inclusion. Thus the unique minimal element $\hat{0}$ of $L(A)$ corresponds to the empty subfamiliy of A and represents the whole space. The poset $L(A)$ is graded by codimension $r(x) := n - \dim x$, especially $r(x \wedge y) + r(x \vee y) \leq r(x) + r(y)$ if the join $x \vee y := x \cap y$ is defined. Remember that the meet

$$x \wedge y := \inf\{z \in L(A) \mid x \cup y \subset z\}$$

is always defined.

The union $\bigcup_{h \in A} h$ has the homotopy type of the geometric realisation $|\Delta L^\bullet|$ of the order-complex $\Delta L^\bullet$ of $L^\bullet$, $L^\bullet := L - \{\hat{0}\}$.

The homotopy type of the compactification

$$N(A) := \bigcup_{x \in A} S_x, S_x := x \cup \{\infty\} \subset S^n = \mathbb{R}^n \cup \{\infty\}$$

is also completely determined by the combinatorical data $(L(A), r, n)$.

Proposition 1 *(Ziegler [10])*

$$N(A) \simeq \bigvee_{x \in L^\bullet} S^{\dim x} * |\Delta L^\bullet_{<x}|.$$

Here $\Delta L^\bullet_{<x}$ is order-complex of the interval $\{y \in L \mid \hat{0} < y < x\}$ and $S^m * \emptyset := S^m$ by convention.

By Alexander duality it follows that the cohomology groups of the complement

$$M(A) := S^n - N(A) = \mathbb{R}^n - \bigcup_{h \in A} h$$

of the arrangement $L(A)$ are combinatorically determined as well.

Corollary 1 *(Goresky-MacPherson [3])*

$$\tilde{H}_k(M(A), \mathbb{Z}) = \bigoplus_{x \in L^\bullet} \tilde{H}^{r(x)-k-2}(|\Delta L^\bullet_{<x}|, \mathbb{Z}).$$

The Möbius function for the poset $L(A)$ is recursively defined for $x \leq y$ by

$$\mu(x,x) = 1, \quad \sum_{x \leq z \leq y} \mu(x,z) = 0 \quad \text{for} \quad x < y$$

and it is well known that the Euler characteristic satisfies

$$\chi(|\Delta L^\bullet_{<x}|) = \mu(0,x) + 1.$$

Now if $L(A)$ is a complex arrangement $r(x) \equiv 0(2)$ hence

Corollary 2

$$\chi(M(A)) = \sum_{x \in L(A)} \mu(0,x).$$

For complex hyperplane arrangements the cohomology-algebra $H^*(M(A), \mathbb{Z})$ is as well combinatorically determined as shown by Orlik-Solomon [7] whereas in the real situation this is no longer true [11]. In fact one can show the following: Let $A_{4,4}$ be the moduli space (resp. simultaneous linear equivalence) of all central arrangements of four 2-dimensional subspaces $h_1, \ldots, h_4 \subset \mathbb{R}^4$ suche that $h_i \cap h_j = 0$ for $i \neq j$. Clearly $(L(A), r, 4) \cong (L(A'), r', 4)$ for all $A, A' \in A_{4,4}$.

Proposition 2 *(Batzies [1])*

(1) $A_{4,4}$ *has exactly two path-components;*
(2) $H^*(M(A), \mathbb{Z}) \ncong H^*(M(A'), \mathbb{Z})$ *if* A, A' *are in different path-components;*
(3) $M(A) \approx M(A')$ *if* A, A' *are in the same path-component.*

On the other hand by Spanier-Whitehead duality the stable homotopy type of $M(A)$ *is combinatorically determined, i.e.* $\sum^m M(A) \simeq \sum^m M(A')$ *for* $m \gg 0$ *if* $(L(A), r, n) \cong (L(A'), r', n')$.

Example: Consider the two arrangements A and A' in $\mathbb{R}^2$ consisting of a line and two points either on one side or on different sides of the line. Then

$$M(A) \simeq S^1 \amalg S^1 \not\simeq M(A') \simeq S^1 \vee S^1 \vee S^0 \ ,$$

but

$$\sum M(A) \simeq \sum M(A') \simeq S^2 \vee S^2 \vee S^1.$$

In the late eighties Orlik [6] conjectured that for complex hyperplane arrangements the homotopy-type of $M(A)$ is combinatorically determined. In 1994 Rybnikov [8] sketched a counterexample in $\mathbb{C}^2$ by studying the lower central series of the fundamental group $\pi_1(M(A))$.

Theorem 1 *(Schaper [9]) Let A be a ≥ 2-arrangement in $\mathbb{R}^n$, i.e. $\dim x - \dim y \geq 2$ for all $x, y \in L(A)$, $x < y$. Then the homotopy type of the first suspension $\sum M(A)$ is combinatorically determined.*

Corollary 3 *For a complex arrangement the first suspension of the complement is combinatorically determined.*

Remark 1 *If A is an arrangement in $\mathbb{R}^n$ let A^+ the arrangement in $\mathbb{R}^{n+1}$ defined by $h \times 0$, $h \in A$. Then $\sum M(A) \simeq M(A^+)$.*

Let $\tau_A := \max\{0, 2\dim \Delta L^\bullet_{\leq x} - 2 - r(x) \mid x \in L^\bullet\}$. Then the homotopy type of $\sum^{\tau_A+2} M(A)$ is combinatorically determined.

Crucial for the proof of Schaper's theorem is the following: For each $x \in L^\bullet$ there is a PL-sphere $T_x^\# \subset S^n$ of dimension $r(x) - 1$ and a PL-embedding

$$\Psi_x : |\Delta L^\bullet_{<x}| \longrightarrow T_x^\#$$

such that

$$\text{(1)} \qquad S^n \approx S_x * T_x^\# \ ,$$

$$\text{(2)} \qquad \Psi_x[x_1, \ldots, x_s] \subset T_x^\# \cap S_{x_1} \quad \text{for all chains} \quad x_1 < \cdots < x_s \quad \text{in} \quad L^\bullet_{<x}.$$

These mappings are constructed by Whitney's embedding theorem and for two different Ψ_x's the complements $T_x^\#$-image Ψ_x are homotopically equivalent. The second condition guaranties that the mappings Ψ_x fit to a homotopy equivalence

$$\Gamma := \bigvee_{x \in L^\bullet} S^{\dim x} * |\Delta L^\bullet_{<x}| \xrightarrow{\simeq} N(A).$$

Since each Ψ_x is an embedding there is an embedding $\Gamma \subset S^{n+1}$ and a mapping cylinder C for

$$\Gamma \xrightarrow{\simeq} N(A) = N(A^+)$$

in S^{n+1}, i.e. a deformation retraction

$$C \longrightarrow N(A^+)$$

such that

$$\begin{array}{ccc} \Gamma & \subset & C \\ \downarrow\simeq & \swarrow & \\ N(A^+) & & \end{array}$$

commutes. Furthermore the complements of $N(A^+)$ as well as those of Γ and C have the homotopy type of simply connected CW-spaces, hence by Alexander duality and Whitehead's theorem $\sum M(A) \simeq M(A^+) \simeq S^{n+1} - \Gamma \simeq \sum \bigvee_{x \in L^\bullet} (T_x^\#\text{-image } \Psi_x)$.

Remark 2 *In general the homotopy type of* $M(A)$ *and* $\Gamma^\# := \bigvee_{x \in L^\bullet} (T_x^\#\textit{-image } \Psi_x)$ *differ, e.g. for the Boolean arrangement* $z_1 z_2 = 0$ *in* $\mathbb{C}^2$, $M(A) \simeq S^1 \times S^1$ *whereas* $\Gamma^\# \simeq S^1 \vee S^1 \vee S^2$.

Let $r(A) := \min\{r(x) \mid x \in A\}$. *Then* $M(A)$ *is* $(r(A)-2)$*-connected, hence for* ≤ 2*-arrangements* $\pi_k M(A) \simeq \pi_k \Gamma^\#$ *for* $0 \leq k \leq 2.r(A) - 4$ *by the suspension theorem. On the other hand if* $\max\{r(x) \mid x \in A\} \geq 3$ *then infinite many homotopy groups of* $M(A)$ *are non-trivial.*

Finally we will study $\Delta L^\bullet$ from an algebraic point of view. To do this we consider for a finite simplicial complex Δ with vertex set $E = \{x_1, \ldots, x_m\}$ the Stanley-Reisner ring

$$\mathbb{C}[\Delta] = \mathbb{C}[x_1, \ldots, x_m]/I_\Delta$$

and its Lie algebra $\mathbb{D}_\Delta$ of derivations. Here I_Δ is the ideal generated by monomials $x_{i_1} \cdot \ldots \cdot x_{i_s}$, $1 \leq i_1 < \cdots < i_s \leq m$ such that $\{x_{i_1}, \ldots, x_{i_s}\}$ is not a simplex of Δ.

Proposition 3 *(Brumati-Simis [2])*

$$\mathbb{D}_\Delta = \bigoplus_{x \in E} (I : (I : x))/I \cdot \frac{\partial}{\partial x}.$$

If $\mathbb{D}$ *is the Lie algebra of derivations of the full polynomial ring the derivations* $D_x := x \frac{\partial}{\partial x} \bmod I_\Delta.\mathbb{D}$, $x \in E$ *generate a maximal abelian subalgebra* H *of* $\mathbb{D}_\Delta$ *and the* $\mathbb{C}$*-dual* H^* *of* H *may be identified with* $\mathbb{C}^m$ *via* $\lambda \longmapsto (\lambda(D_x))_{x \in E}$. *For* $\lambda \in H^*$ *let*

$$\mathbb{D}_\lambda := \{\ D \in \mathbb{D}_\Delta \mid [h, D] = \lambda(h) D \ \textit{ for all } \ h \in H\ \}.$$

If $\mathbb{D}_\lambda \neq 0$, λ *is said to be a root.*

Proposition 4 *(Müller [5])*

$$\mathbb{D}_\Delta = \bigoplus_{\lambda \in H^*} \mathbb{D}_\lambda.$$

Corollary 4 $\mathbb{D}_\lambda \neq 0$ *iff one of the following three cases occur*

(i) $\lambda = 0$,

(ii) $\lambda \in \mathbb{N}^m - \{0\}$ *and* $supp\lambda := \{x \mid \lambda(D_x) \neq 0\} \in \Delta$,

(iii) $\lambda(D_x) = -1$ *for exactly one* $x \in E$, $\lambda(D_{x'}) \in \mathbb{N}$ *for* $x \neq x'$ *and* $supp(\lambda + \delta_x) \in ast(x) - \Delta_x$,

where $st(\sigma) := \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$ *is the star,* $ast(x) := \{\sigma \in \Delta \mid x \notin \sigma\}$ *the anti-star,* $\Delta_x := \{\sigma \in \Delta \mid st(\sigma) \not\subset st(x)\}$ *and* δ_x *the Kronecker function.*

If all roots λ *are non-negative, the complex is said to be closed.*

Theorem 2 *(Müller [5]) The following statements for closed complexes are equivalent*

(i) $\Delta \cong \Delta'$,

(ii) $\mathbb{D}_\Delta \cong \mathbb{D}_{\Delta'}$.

Proposition 5 *([4]) Let* Δ *be a closed complex such that* $\mathbb{C}[\Delta]$ *is Cohen-Macaulay, i.e.*

$$\textit{depth}\ \mathbb{C}[\Delta] = \dim \Delta + 1.$$

Then $\mathbb{D}_\Delta$ *is a Cohen-Macaulay module.*

By a simple argument based on the Möbius function one gets that the Folkman complex of a geometric lattice is closed.

Corollary 5 *Let* L *be a finite geometric lattice and* $F := \Delta(\hat{0}, \hat{1})$ *the Folkman complex of* L*. Then* $\mathbb{D}_F$ *is a Cohen-Macaulay module.*

If A is a central hyperplane arrangement then $L(A)$ is a finite geometric lattice and

$$|F| \simeq \bigvee\nolimits_m S^k$$

with $m = \mu(\hat{0}, \hat{1})$ and $k = r(\bigcap_{h \in A} h)$, especially the associated module of derivations is Cohen-Macaulay.

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