Some Remarks about the Complete Integrability in the Sense of Arnold-Liouville

Cristian Lăzureanu

Abstract

We give a Lax formulation of some particular Hamiltonian mechanical system and prove their complete integrability.

Let (M, ω, H) be an Hamiltonian mechanical system where:

$$M = \mathbf{R}^{2n}$$

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq^k \tag{1}$$

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n} V_k(q^k)$$

It is clear that it has a separable potential

$$V(q^1, q^2, ..., q^n) = \sum_{k=1}^{n} V_k(q^k)$$

and moreover H can be written in the followings form

$$H = \sum_{k=1}^{n} H_k,$$

where

$$H_k = \frac{1}{2}p_k^2 + V_k(q^k), \quad k = \overline{1, n}.$$

Then the equations of motion are given as usually by Hamilton's equations, [2]:

$$\begin{cases} \dot{q}^k = \frac{\partial H}{\partial p_k} = p_k \\ \\ \dot{p}_k = -\frac{\partial H}{\partial q^k} = -V_k' \end{cases} \quad k = \overline{1, n} \tag{2}$$

where $V_k' = \frac{dV_k}{dq^k}$, $k = \overline{1, n}$.

Theorem 1 *Let us take*

$$V_k(q^k) = \frac{1}{2}U_k^2(q^k), \quad k = \overline{1, n}.$$

Then we have:
(i) If $U_k' = \frac{dU_k}{dq^k} \not\equiv 0$, *for each* $k = \overline{1,n}$, *then the system (1) has the Lax representation* $\dot{L} = [L, B]$, *where*

$$L = \begin{bmatrix} p_1 & U_1 & 0 & 0 & \dots & 0 & 0 \\ U_1 & -p_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & p_2 & U_2 & \dots & 0 & 0 \\ 0 & 0 & U_2 & -p_2 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & p_n & U_n \\ 0 & 0 & 0 & 0 & \dots & U_n & -p_n \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \frac{1}{2}U_1' & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2}U_1' & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}U_2' & \dots & 0 & 0 \\ 0 & 0 & -\frac{1}{2}U_2' & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2}U_n' \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{2}U_n' & 0 \end{bmatrix}$$

(ii) If there exists $k \in \{1, 2, \dots, n\}$ *such that* $U_k' \equiv 0$, *then the system (1) has the Lax representation* $\dot{L} = [L, B]$, *where*

$$L = \begin{bmatrix} p_1 & U_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ U_1 & -p_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & p_k & iU_k & \dots & 0 & 0 \\ 0 & 0 & \dots & -iU_k & p_k & \dots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & 0 & \dots & p_n & U_n \\ 0 & 0 & \dots & 0 & 0 & \dots & U_n & -p_n \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \frac{1}{2}U_1' & \dots & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2}U_1' & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{1}{2}U_n' \\ 0 & 0 & \dots & 0 & 0 & \dots & -\frac{1}{2}U_n' & 0 \end{bmatrix}$$

and $i^2 = -1$.

Proof.
(i) Let us denote

$$L_k = \begin{bmatrix} p_k & U_k \\ U_k & -p_k \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 & \frac{1}{2}U'_k \\ -\frac{1}{2}U'_k & 0 \end{bmatrix}, \quad k = \overline{1,n}.$$

Then $L = diag(L_1, L_2, ..., L_n)$, $B = diag(B_1, B_2, ..., B_n)$ and

$$\begin{aligned} \dot{L} &= diag(\dot{L}_1, \dot{L}_2, ..., \dot{L}_n) \\ [L, B] &= LB - BL \\ &= diag(L_1B_1 - B_1L_1, L_2B_2 - B_2L_2, ..., L_nB_n - B_nL_n) \\ &= diag([L_1, B_1], [L_2, B_2], ..., [L_n, B_n]) \end{aligned}$$

Therefore $\dot{L} = [L, B]$ if and only if $\dot{L}_k = [L_k, B_k]$, $k = \overline{1,n}$.
The system (1) has a Lax representation if its equations of motion are equivalent with the equations $\dot{L} = [L, B]$. For this, it is sufficient to prove that

(3) $$\begin{cases} \dot{q}^k = p_k \\ \dot{p}_k = -V'_k \end{cases} \Leftrightarrow \dot{L}_k = [L_k, B_k], \quad k = \overline{1,n}.$$

We have:

$$\dot{L}_k = \begin{bmatrix} \dot{p}_k & \dot{U}_k \\ \dot{U}_k & -\dot{p}_k \end{bmatrix} = \begin{bmatrix} \dot{p}_k & U'_k \cdot \dot{q}^k \\ U'_k \cdot \dot{q}^k & -\dot{p}_k \end{bmatrix}$$

$$[L_k, B_k] = L_kB_k - B_kL_k = \begin{bmatrix} -U_kU'_k & p_kU'_k \\ p_kU'_k & U_kU'_k \end{bmatrix} = \begin{bmatrix} -V'_k & p_kU'_k \\ p_kU'_k & V'_k \end{bmatrix}$$

and taking into account that $U'_k \not\equiv 0$ we obtain the desired result.
(ii) If $j \neq k$ the equivalence (3) holds obviously. If $k \in \{1, 2, ..., n\}$ and $U'_k \equiv 0$ we shall take

$$L_k = \begin{bmatrix} p_k & iU_k \\ -iU_k & p_k \end{bmatrix}, \quad B_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and then the equations (2) become:

(4) $$\begin{cases} \dot{q}^k = p_k \\ \dot{p}_k = 0 \end{cases}$$

On the other hand,

$$\dot{L}_k = \begin{bmatrix} \dot{p}_k & 0 \cdot \dot{q}^k \\ 0 \cdot \dot{q}^k & \dot{p}_k \end{bmatrix}; \quad [L_k, B_k] = L_kB_k - B_kL_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

and then our assertion follows immediately.■

Remark 1 *An easy computation shows us that*

$$H = \frac{1}{4}Trace(L^2).$$

Remark 2 *In the proof. of* **Theorem 1**, *U_k is choosen in the following manner:*
a) $U_k(q^k) = \sqrt{2V_k(q^k)}$, *if* $V_k(q^k) > 0$;
b) $U_k(q^k) = i\sqrt{-2V_k(q^k)}$, *if* $V_k(q^k) < 0$.
c) If $V_k(q^k)$ *has some roots* $\alpha_1, ..., \alpha_p$, $\alpha_1 < ... < \alpha_p$ *we shall restrict our considerations to the intervals* $(-\infty, \alpha_1)$, (α_1, α_2), ...,(α_p, ∞).

Remark 3 *The Lax formulation given in* **Theorem 1** *is useful in the study of complete integrability of the system (1) since, from Flaschka's theorem ([1]), the eigenvalues of L are time-independent. Using the form of L it follows that these eigenvalues are in fact the eigenvalues of the matrices* L_k, $k = \overline{1,n}$.

Theorem 2 *The system (2) is completely integrable on the set*

$$D = \mathbf{R}^{2n} \setminus \{(q^1, ..., q^n, p_1, ..., p_n) \in \mathbf{R}^{2n} |$$
$$A_1 \cdot A_2 \cdot ... \cdot A_k \cdot ... \cdot A_n = 0, \text{ where } A_k = p_k \text{ or } A_k = V_k', \quad k = \overline{1,n}\}.$$

Proof. We need n constants of motion in involution and functionally independent on the set D. Let us take

$$K_0 = H, K_1 = H_1, ..., K_{n-1} = H_{n-1}.$$

Then, for $i \neq j$, $i, j \in \{1, 2, ..., n-1\}$ it follows that:

$$\begin{aligned} \{K_i, K_j\} &= \{H_i, H_j\} \\ &= \sum_{k=1}^{n} \left(\frac{\partial H_i}{\partial q^k} \cdot \frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k} \cdot \frac{\partial H_j}{\partial q^k}\right) \\ &= 0 \end{aligned}$$

If $j \in \{1, 2, ..., n-1\}$ then we have:

$$\{K_0, K_j\} = \{H, H_j\} = \{\sum_{k=1}^{n} H_k, H_j\} = \sum_{k=1, k\neq j}^{n} \{H_k, H_j\} + \{H_j, H_j\} = 0.$$

Therefore $K_0, ..., K_{n-1}$ are in involution. $K_0, K_1, ..., K_{n-1}$ are functionally independent if rank $A = n$ on the set D, where

$$A = \left[\frac{\partial K_j}{\partial q^k}, \frac{\partial K_j}{\partial p_k}\right]_{j=\overline{0,n-1},\ k=\overline{1,n}}$$

We have

$$A = \begin{bmatrix} V_1' & V_2' & \dots & V_{n-1}' & V_n' & p_1 & p_2 & \dots & p_{n-1} & p_n \\ V_1' & 0 & \dots & 0 & 0 & p_1 & 0 & \dots & 0 & 0 \\ 0 & V_2' & \dots & 0 & 0 & 0 & p_2 & \dots & 0 & 0 \\ \vdots & & & & & & & & & \\ 0 & 0 & \dots & V_{n-1}' & 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

and the minors of order n of A are exactly the products from D.■

Example 1 The n-dimensional harmonic oscillator is a Hamiltonian mechanical system (M, ω, H), where $M = \mathbf{R}^{2n}$, $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$,

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}(q^k)^2$$

having the Hamilton equations:

$$\begin{cases} \dot{q}^k = p_k \\ \dot{p}_k = -q^k \end{cases} \qquad k = \overline{1, n}.$$

Then $V_k(q^k) = \frac{1}{2}(q^k)^2$ and $U_k = q^k$ with $U_k' = 1$. We obtain the Lax formulation $\dot{L} = [L, B]$ with

$$L = \begin{bmatrix} p_1 & q^1 & \dots & 0 & 0 \\ q^1 & -p_1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & p_n & q^n \\ 0 & 0 & \dots & q^n & -p_n \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \frac{1}{2} & \dots & 0 & 0 \\ -\frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & \frac{1}{2} \\ 0 & 0 & \dots & -\frac{1}{2} & 0 \end{bmatrix}$$

Considering the constants of motion $K_0 = H, K_1 = H_1, ..., K_{n-1} = H_{n-1}$, where

$$H_k = \frac{1}{2}p_k^2 + \frac{1}{2}(q^k)^2,$$

we obtain that the n-dimensional harmonic oscillator is completely integrable on the set

$$D = \mathbf{R}^{2n} \setminus \{(q^1, ..., q^n, p_1, ..., p_n) \in \mathbf{R}^{2n} |$$
$$A_1 \cdot A_2 \cdot ... \cdot A_k \cdot ... \cdot A_n = 0, \text{ where } A_k = q^k \text{ or } A_k = p_k, \quad k \in \{1, ..., n\} \}.$$

References

[1] H. Flaschka, *On the Toda lattice*, Phys.Rev. B9(1974), 1924-1925.

[2] M. Puta, *Hamiltonian Mechanical System and Geometric Quantization*, Kluwer Academic Publishers, 1993.

Cristian Lăzureanu
Department of Mathematics
"Politehnica" University
Piaţa Horaţiu No 1
1900 Timişoara, Romania