The Diffeomorphism Group of a Riemannian Foliation

E. Macias Virgós[1]

Abstract

We present without proofs some recent results on Riemannian foliations. First we consider the (infinite dimensional) manifold $\mathcal{M}(M/F)$ of bundle-like metrics on a foliated manifold (M, F). Then we show how to construct an adapted local addition for any Riemannian foliation. Also the notion of foliate vector field along a map is introduced. This allows us to endow the space $\mathrm{Diff}(M/F)$ of foliation preserving diffeomorphisms with a manifold structure. Finally, we study the group of transverse isometries of a Lie foliation in the setting of Q-manifolds.

1 Introduction

In this paper we present without proofs some recent results on Riemannian foliations. General references for this subject are the book of P. Molino [11] and the survey of Ph. Tondeur in [19].

Example 1 *A linear flow F on the torus T^2.* The leaves of such a foliation (=completely integrable distribution) are lines of some constant slope α. When α is a rational number, F is a trivial bundle with compact fibres. When α is not rational, F is no longer a fibre bundle, but still has some nice geometrical properties, because there is a Riemannian metric g on T^2 such that the leaves remain at the same distance.

Definition 2 *The foliation F is said to be Riemannian if there exists some Riemannian metric g on the ambient manifold M which is* bundle-like*, that is the transverse part of the metric is constant when we move along the leaves.*

If (x^i, y^α), $i = 1, \ldots, m$, $\alpha = 1, \ldots n$, denote the tangent and transverse coordinates of an adapted chart, then a bundle–like metric has the form

$$\begin{pmatrix} * & * \\ * & g_{\alpha\beta}(y) \end{pmatrix}.$$

Example 3 *Lie foliations* are the simplest examples of Riemannian foliations. Let M be a compact manifold. Let us consider the kernel of some 1-form $\omega_x \colon T_xM \to \mathcal{G}$ of maximal rank with values in the Lie algebra $\mathcal{G}$ of a connected simply connected Lie group G. The integrability condition becomes $d\omega + (1/2)[\omega, \omega] = 0$. It is a result of E. Fédida that for such a G-Lie foliation there exists some representation $h \colon \pi_1(M) \to G$ and some regular covering $\pi \colon \tilde{M} \to M$ with group of deck transformations $\Gamma = \mathrm{im} h$, such that:

[1] *Partially supported by DGICYT research project (Spain).*

1. the lifted foliation π^*F is a locally trivial bundle $D\colon \tilde{M} \to G$;
2. it is h-equivariant: $D(\gamma(p)) = D(p)h(\gamma)$.

Moreover, leaves are dense in M if and only if Γ is a dense subgroup of G.

For instance, a linear flow in T^2 is defined by the closed 1-form $\omega = dy - \alpha dx$. By computing the group of periods of such a form we obtain the group morphism $h\colon \pi_1(T^2) \to R$,

$$h([\alpha]) = \int_\alpha \omega,$$

with image $\Gamma = \{n + \alpha m \in R \mid n, m \in Z\}$. Then, the lifted foliation is the foliation on R^2 whose leaves are the fibres of the (trivial) bundle $D\colon R^2 \to R$, $D(x,y) = y - \alpha x$.

The structure of Riemannian foliations on compact manifolds is well known. Let us consider the $SO(n)$-bundle $\pi\colon E \to M$ of transverse orthonormal frames for a bundle-like metric. Then, P. Molino proved [11] that the closures of the leaves of a lifted foliation are the fibres of some locally trivial bundle $E \to B$, while the induced foliation on each closure is a Lie foliation. Then, Lie foliations with dense leaves are the simplest but fundamental blocks for constructing Riemannian foliations.

Also, in [16] A. Haefliger and E. Salem considered the transverse holonomy pseudogroup $\mathcal{P}$ of the foliation. To be Riemannian means that the elements of $\mathcal{P}$ are isometries. From this point of view, Molino's theorem is a generalization of the Myers-Steenrod theorem on the Lie group of isometries of a Riemannian manifold.

2 Infinite dimensional manifolds

In this section I shall discuss some of my joint work with E. Sanmartín (Univ. of Vigo, Spain). We are interested in the following global objects associated to a Riemannian foliation F on the manifold M (compact or not):

1. The (infinite dimensional) manifold $\mathcal{M}(M/F)$ of all possible bundle-like metrics for F;
2. The group $\mathrm{Diff}(M/F)$ of foliation preserving diffeomorphisms.
3. The action of $\mathrm{Diff}(M/F)$ on $\mathcal{M}(M/F)$.

This setting can be viewed as a generalization of some classical results about the space of Riemannian metrics $\mathcal{M}(M)$ and the group $\mathrm{Diff}(M)$ of diffeomorphisms of a Riemannian manifold M (that is $F =$ foliation by points).

For instance, D. G. Ebin [3] proved in 1970 that $\mathcal{M}(M)$ is an infinite dimensional manifold and endowed it with a metric which is invariant under the action of $\mathrm{Diff}(M)$. This seminal work has been generalized in many ways. Let us cite L. M. Răileanu [14], who considered manifolds with boundary, Freed and Groisser [4], who made a geometrical study of the manifold $\mathcal{M}(M)$, and Gil and Michor [5], who solved the non–compact case.

On the other hand, for the group $\mathrm{Diff}(M)$ we have the well known works of Palais [13], Omori [12] and Leslie [7] among many other authors.

2.1 The space of bundle-like metrics

Let F be a Riemannian foliation on a (not neccesarily compact) manifold M. We know rather well the structure of the space $\mathcal{M}(M/F)$ of all possible bundle-like metrics. In fact, E. Sanmartín has recently generalized in [17] many of the results about $\mathcal{M}(M)$, by considering any oriented vector bundle ξ and the manifold $\mathcal{M}(\xi)$ of the so–called metric sections. The classical results correspond to $\xi = TM$. For instance, she defines on $\mathcal{M}(\xi)$ a natural connection and she computes the geodesics and curvature. Also she considers the (trivial) bundle over the manifold of volume forms.

By particularizing these results to the tangent and normal subbundles of the foliation, she is able to prove that $\mathcal{M}(M/F)$ is a closed subspace of $\mathcal{M}(M)$, which inherits a structure of infinite dimensional manifold. This work has been used later by J. Alvarez and Y. Kordyukov [2] who proved that $\mathcal{M}(M/F)$ is a deformation retract of $\mathcal{M}(M)$. Other related questions appear as open problems in the Proceedings of the Workshop on Foliations held at Santiago de Compostela in 1994 [9].

2.2 Foliation preserving diffeomorphisms

Let $\mathrm{Diff}(M/F)$ be the group of diffeomorphisms of M that preserve a Riemannian foliation F. More generally, we can consider two foliated manifolds (N, F') and (M, F) and the space $\mathcal{C}^\infty_F(N, M)$ of smooth maps sending leaves into leaves. In order to endow these spaces with a structure of manifold we must acomplish several objectives.

(a) First, we put on $\mathrm{Diff}(M/F)$ some locally path connected topology, namely the $\mathcal{C}^\infty$ Whitney's topology (in the compact case) or the $\mathcal{FD}$–topology of Michor in the non compact case [8].

(b) Second, we must have some local model for our manifold. In the classical case of $\mathrm{Diff}(M)$ this model is the space of vector fields on M. For the foliated case, we consider the space $\chi_c(M/F)$ of foliate vector fields (with compact support), that is vector fields X whose flow preserves the foliation.

More generally, let $f \in \mathcal{C}^\infty_F(N, M)$. Then a vector field along f is any smooth section X of the pull-back bundle f^*TM. If λ is a smooth function defined in $V \subset M$, let $X\lambda \in \mathcal{C}^\infty(f^{-1}(V)$ be the function given by

$$(X\lambda)(x) = (d\lambda)_{f(x)}(X_{f(x)}.$$

We say that X is foliate when the latter action preserves basic functions, that is functions which are constant along the leaves.

Proposition 4 *The space $\Gamma^F_c(f^*TM)$ of foliate vector fields along f is a LF–topological vector space, that is an inductive limit of Fréchet spaces (a Fréchet space in the compact case).*

The proof is rather technical, but essentially consists in proving that the space of basic functions is a closed subspace of $\mathcal{C}^\infty(M)$, hence $\Gamma^F_c(f^*TM)$ is closed in a space of sections. When f is a diffeomorphism, we have $\Gamma^F_c(f^*TM) \equiv \chi_c(M/F)$.

(c) Now we must construct an atlas $\varphi_f\colon \mathcal{U}_f \to \Gamma_c^F(f^*TM)$, where $\mathcal{U}_f$ is some neighbourhood of f, and the local model $\Gamma_c^F(f^*TM)$ depends on f (as typically occurs in the infinite dimensional manifolds).

We follow the technique of P. Michor as in [10].

Definition 5 *A local addition on the manifold M is any smooth map*

$$E\colon \Omega \subset TM \to M,$$

defined in some open neighbourhood Ω of the zero section of the tangent bundle $\pi\colon TM \to M$, such that

1. *$(\pi, E)\colon \Omega \to M \times M$ is a diffeomorphism onto an open neighbourhood Ω' of the diagonal;*
2. *$E(0_p) = p$ for all $p \in M$.*

In a Riemannian manifold the exponential map defines a local addition.

Then, a chart around f for the manifold $\mathrm{Diff}(M)$ will be defined on the set

$$\mathcal{U}_f = \{g \mid (f(x), g(x)) \in \Omega' \text{ for all } x \in M\}$$

(in the non-compact case, f and g must be equal outside some compact set), while $\varphi_f(g)$ is the only vector field X along f such that

$$(\pi, E)(X_x) = (f(x), g(x)).$$

For a foliated manifold the flow of this vector field X does not preserve the foliation. So we need to have a local addition which is adapted to the foliation, in the sense that it induces another one in a small transversal of each leaf. More precisely,

Definition 6 *We say that the local addition E is adapted to the foliation if there exists some covering $\{U_i\}$ of M by adapted charts $\pi_i\colon U_i \to \overline{U}_i$ such that for each $U = U_i$ we have:*

1. *The restriction of E to TU is a local addition $E_U\colon \Omega_U \subset TU \to U$;*
2. *it induces a local addition $\overline{E}_U\colon (\pi_U)_*(\Omega_U) \subset T\overline{U} \to \overline{U}$ such that $\pi_U E_U = \overline{E}_U(\pi_U)_*$*

Proposition 7 *Let E be an adapted to the foliation local addition. Then the chart $(\mathcal{U}_f, \varphi_f)$ defined above verifies that a map g sends leaves into leaves iff the vector field $X = \varphi_f(g)$ is foliate.*

2.3 The adapted local addition

In this section we will sketch how to obtain an adapted local addition in a Riemannian foliation.

Any foliation of codimension n can be defined by some collection of submersions $\pi_U : U \to \overline{U} \subset R^n$ whose domains cover the manifold. To be Riemannian means that the maps π_U are Riemannian submersions and the transverse changes of coordinates $\pi_V = \gamma_{UV}\pi_U$ are isometries.

Example 8 Let F be a G-Lie foliation as in Example 3. Let g be any Riemannian metric on M, orthogonally decomposed as $g = g_F + g_\nu$, that we lift to a Γ-invariant metric $\tilde{g} = \tilde{g}_F + \tilde{g}_\nu$ on the covering $\widetilde{M}$. Now, we can change the transverse part $\tilde{g}_\nu$ to some right invariant metric on the Lie group G. In this way, D is a Riemannian submersion and the transverse changes of coordinates are translations by Γ, thus isometries.

Let g be a bundle-like metric on the foliated manifold (M, F). Since the leaves may not be totally geodesic submanifolds, we must clearly distinguish the geodesics in the manifold and the geodesics in each leaf L for the induced metric g^L. The main property we need is [15] that if a geodesic γ is orthogonal in some point to a leaf, then it is also orthogonal to all leaves encountering it.

Then, for a sufficiently small adapted neighbourhood W of each point $x \in M$ and for vectors $X \in T_xM$ with $\mid X \mid < \varepsilon$, we define $E(v)$ as follows.

First, we orthogonally decompose $X = T + H$ in its tangent and transverse parts. Then we consider the point $y = \exp_x^L(T)$ which is in the same plate than x, and we lift the horizontal vector H_x to some horizontal vector H_y by using that W is the domain of a Riemannian submersion onto the transverse submanifold $\overline{W}$. Finally, we take $E(X) = \exp_y(H_y)$.

We remark that this definition does not depend on W because the transverse change of coordinates are isometries. Finally, E is a local addition because for any point z near x we have a horizontal geodesic γ connecting z with some point y in the plate of x, and we can take a geodesic γ^L in the leaf joining x to y. Then $z = E(T + H_x)$ for $T = (\gamma^L)'(0)$ and $H_y = -(\gamma)'(0)$.

3 Transverse isometries

One of Ebin's results [3] about the space of Riemannian metrics is that most metrics have no isometries. This is a consequence of the existence of 'slices' for the action of the group of diffeomorphisms.

Let f be a diffeomorphism which preserves the foliation, g a bundle-like metric. Then the metric f^*g is bundle-like too. We only have partial results in the study of this action of $\mathrm{Diff}(M/F)$ on $\mathcal{M}(M/F)$, so we begin by examining the isotropy group of the metric g, that is the group $\mathrm{Iso}(M/F, g)$ of preserving foliation g–isometries. Since it is hard to control the tangent part of an isometry, we try to understand the structure of the transverse isometries (as a matter of fact, many interesting properties of foliations are transverse).

3.1 Q-manifolds

We focus our attention on Lie foliations with dense leaves. As a consequence of Fédida's results, the space of leaves is the (topologically trivial) space G/Γ, that we can consider as a Q–manifold in the sense of R. Barre [1]. The author and G. Hector (University of Lyon, France) have proven some results in this direction [6].

Let X be a manifold, $\mathcal{P}$ a pseudogroup of local diffeomorphisms of X whose domains recover X. The quotient space $X/\mathcal{P}$ is called a Q-manifold when the following condition is satisfied:

Q-condition: Let $c, c' : T \to X$ be two smooth maps which are $\mathcal{P}$-related —that is, for all $y \in T$ there exists $\gamma_t \in \mathcal{P}$ such that $c'(t) = \gamma_t c(t)$. If $c(t_0) = c'(t_0)$ then $c = c'$ in some neighbourhood of t_0.

As G. Meigniez has pointed out, one can suppose without loss of generality that c, c' are smooth paths defined on some interval $T = (-\epsilon, +\epsilon)$.

The map $f: X/\mathcal{P} \to X'/\mathcal{P}'$ between Q-manifolds is said to be *smooth* if for any class $\pi(x) \in X/\mathcal{P}$ there exists an open neighbourhood U of some representative x, and a smooth map $f_U: U \to X'$, such that $\pi' f_U = f\pi$. Obviously, for any other representative $\gamma(x)$ we have the map $f_U \gamma^{-1}$; moreover smoothness implies continuity.

Then f is determined by the family $\{f_U\}$ of local lifts, which verify:

1. The $\mathcal{P}$-saturated of the open sets U cover X;
2. If $x \in U \cap \gamma V$ for some $\gamma \in \mathcal{P}$, then there exists $\gamma' \in \mathcal{P}'$ such that $f_U = \gamma' f_V \gamma^{-1}$ in some neighbourhood of x.

Proposition 9 *The map f is a diffeomorphism if and only if the family $\{f_U\}$ of local lifts also verifies:*

1. *each f_U is a local diffeomorphism;*
2. *the saturaded of the open sets $f_U(U)$ cover X'.*

Theorem 10 *Assume the manifold X is connected and simply connected. Then any diffeomorphism $f: X/\mathcal{P} \to X'/\mathcal{P}'$ can be lifted to some local diffeomorphism $f_X : X \to X'$. If moreover X' is connected and simply connected then f_X is a global diffeomorphism.*

3.2 Riemannian Q-manifolds

The differentials $T\gamma$ of the elements of $\mathcal{P}$ define a pseudogroup $T\mathcal{P}$ acting on the tangent bundle TX in such a way that $TX/T\mathcal{P}$ is again a Q-manifold, which we call $T(X/\mathcal{P})$.

A vector field in $X/\mathcal{P}$ is then a smooth section of the tangent bundle $T(X/\mathcal{P}) \to X/\mathcal{P}$, that is a family ξ_U of local vector fields on X such that $\xi_U(x) = d\gamma'_y(\xi_V(y))$ for some γ' when $x = \gamma(y) \in U \cap \gamma V$; but this implies $x = \gamma'(y)$, so $\gamma = \gamma'$ near y. Then ξ is a *global* vector field on X which is invariant by the action of $\mathcal{P}$.

The differential of a smooth map $f: X/\mathcal{P} \to X'/\mathcal{P}'$ is the map

$$Tf: T(X/\mathcal{P}) \to T(X'/\mathcal{P}')$$

whose local lifts are $Tf_U: TU \to TX'$.

A Riemannian metric on the Q-manifold $X/\mathcal{P}$ is a Riemannian metric g in X for which the elements of $\mathcal{P}$ are isometries, that is $\gamma^* g = g$ for all $\gamma \in \mathcal{P}$. This is equivalent to saying that g is a smooth section of the bundle $S^2T(X/\mathcal{P})^*$ of symmetric bilinear forms on the tangent bundle (recall that vector fields are global).

If such a g exists, the Q-manifold is said to be Riemannian.

3.3 Lie foliations

Let g be a right invariant Riemannian metric on G. We call $\mathrm{Isom}(G, \Gamma)$ the group of isometries of (G, g) which send classes on classes, that is $\phi(g\Gamma) = \phi(g)\Gamma\ \forall g \in G$. For instance, all isometric left translations L_x, $x \in G$, and right translations R_γ by elements of Γ belong to this group.

Proposition 11 1. *The group R_Γ of right translations by elements of Γ is an invariant subgroup of* $\mathrm{Isom}\,(\mathrm{G}, \Gamma)$.

2. *The group* $\mathrm{Isom}\,(\mathrm{G}/\Gamma)$ *of isometries of the Q-manifold G/Γ is the quotient group* $\mathrm{Isom}\,(\mathrm{G}, \Gamma)/\mathrm{R}_\Gamma$.

As a consequence, $\mathrm{Isom}(G/\Gamma, g)$ is the fibrered product of right translations R_G and a subgroup of the compact Lie group of isometric automorphisms, which can be easily computed.

Example 12 Let us consider the Heisenberg group of real matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Let Γ be the subgroup corresponding to matrices with x, y, z of the form $n + \alpha m$, $n, m \in Z$, for some non rational number α. Then the group of isometries of G/Γ is the semidirect product of $R/\{n + \alpha m\}$ and a finite group of eight elements.

This kind of result can be generalized to any nilpotent simply connected Lie group and a finitely generated dense subgroup Γ, a class of transverse structure for which it is known that there always exists a corresponding Lie foliation on a compact manifold.

References

[1] R. Barre, *De quelques aspects de la théorie des Q-variétés différentielles et analytiques*, Ann. Inst. Fourier (Grenoble) 23 (1973), 227–312.

[2] J.A. Alvarez, Y. Kordyukov, *Long time behaviour of leafwise heat flow for Riemannian foliations*, Preprint 1995.

[3] D. G. Ebin, *The manifold of Riemannian metrics*, Proc. Sympos. Pure Math. 15 (1970), 11-40.

[4] D. Freed, D. Groisser, *The basic geometry of the manifold of Riemannian metrics and of its quotients by the diffeomorphism group*, Michigan Math. J. 36 (1989), 323-344.

[5] O. Gil, P. W. Michor, *The Riemannian manifold of all Riemannian metrics*, Quart. J. Math. Oxford 42 (1991), 183-202.

[6] G. Hector, E. Macias–Virgós, *The group of isometries of a Riemannian Q–manifold*, Preprint 1997.

[7] J. Leslie, *On a differentiable structure for the group of diffeomorphisms,* Topology 6 (1967), 263 - 271.

[8] J. Margalef, E. Outerelo, *Differential Topology*, North Holland Math. Studies Series, 123, 1992.

[9] X. Masa, E. Macias, J. A. Alvarez (Eds.), *Analysis and Geometry in Foliated Manifolds*, World Scientific (1995).

[10] P. W. Michor, *Manifolds of Differentiable Mappings*, Shiva Math. Series, 1980.

[11] P. Molino, *Riemannian Foliations*, Progress in Math. 73, Birkhäuser, Boston (1988).

[12] H. Omori, *Infinite Dimensional Lie Transformation Groups*, Lecture Notes in Math. 427, Springer - Verlag, New - York (1976).

[13] R. S. Palais, *Foundations of Global non-linear Nnalysis*, W. A. Benjamin, New-York (1968).

[14] L.M. Răileanu, *The manifold of Riemannian metrics on a compact manifold with boundary*, An. Stiint. Univ. 'Al. I. Cuza' Iasi. Sect. I Mat. 34 (1988), 67-72.

[15] B. L. Reinhart, *Foliated manifolds with bundle-like metrics,* Annals of Math. 69, no.1 (1959) 119-131.

[16] E. Salem, *Une généralisation du théorème de Myers-Steenrod aux pseudogroupes d'isométries*, Ann. Inst. Fourier (Grenoble) 38 (1988).

[17] E. Sanmartín - Carbón, *The manifold of bundle-like metrics for a Riemannian foliation*, Q. J. Math. Oxford (1997)

[18] Ph. Tondeur, *Foliations on Riemannian Manifolds*, Universitext, Springer-Verlag, New - York (1988).

[19] Ph. Tondeur, *Survey of Riemannian Foliations. In A.C. Albu, M. Craioveanu* (Eds.), Proceedings of the 24-th National Conference of Geometry and Topology, Timisoara (1996).

Dpto. Xeometria e Topoloxia, Fac. Matemáticas,
Univ. de Santiago de Compostela 15706 Galicia (Spain)
macias@zmat.usc.es